MATHEMATICS

PART II

A Textbook for Grade 12

M12TB-II

AUTHORS

Dr. Sumit Kumar Sharma
Assistant Professor
Department of Mathematics, Kirori Mal College, University of Delhi, India

Dr. Gitanjali
Assistant Professor
Department of Applied Sciences, Maharaja Surajmal Institute of Technology, India

REVIEWERS

Professor Isaac Saye-Lakpoh Zawolo
Superintendent
Monrovia Consolidated School System (MCSS)

Charles Tieh Bropleh
Mathematics Specialist
Ministry of Education

Matthew V. Z. Darblo
Mathematics Instructor
University of Liberia (UL)

Ministry of Education
Monrovia, Republic of Liberia

Star Educational Books Distributors (P) Ltd.
Delhi, India

ISBN : 978-93-95626-55-2

Printed on 80 gsm Maplitho paper in Times New Roman 12 pt.
Typeset and Cover designed by Shri Ganpati Enterprises, Delhi - 110 052

Published and Printed at:
Star Educational Books Distributors (P) Ltd., 4736/23, Ansari Road, Darya Ganj, New Delhi - 110002, India for Ministry of Education, Monrovia, Republic of Liberia

Email: info@estar-bk.com, Website: www.estar-bk.com

Foreword

Liberia, having gone through a period of utmost turmoil till 2003, due to the civil wars, is still reeling under its effect and the added trauma of Ebola in 2014 and effects of the COVID-19 outbreak in 2020. The Liberian government, in the past decade, has made valiant efforts to bring order to the lives of its people. In one such effort, the Ministry of Education (MoE) brought changes to the National Curriculum Framework which are relevant to the present generation, and which would prepare them to meet the challenges of the changing trends of the world. The National Curriculum Framework (NCF) 2018 recommends a change in basic assumptions in the teaching learning process from behaviorist to constructivist approach — moving from hardcore print material to the digital world. Keeping in consideration the sociocultural context and varied experiences of learners as laid down in the Framework, our Teaching Learning Materials are expected to be competent to use multiple methods and techniques like e-learning resources, energized textbooks, and readily available reference material to engage the learners.

As a first initiative, the MoE, through its World Bank-funded Improving Results in Secondary Education (IRISE) project, has adapted textbooks for Grades 10 to 12 in five subjects — English Language and Literature, Mathematics, Biology, Physics and Chemistry.

The National Curriculum Framework, 2018, recommends that children's learning at school is a reflection of their life outside the school and shows them the path to become a responsible citizen who makes knowledge-based choices.This principle marks a departure from the legacy of teacher centered learning to student centered learning. The syllabi and textbooks developed on the basis of the NCF indicate a serious attempt to implement the idea of Activity Base Learning (ABL). We hope these measures will take us ahead in the direction of building a system of education as outlined in the NCF.

Combined with the efforts by the school principals and teachers this will encourage children to reflect on their own learning and to pursue imaginative activities and questions.With this in mind, perhaps for the first time in our country, we are able to provide separate subject specific textbooks accompanied with guides for teachers for 10–12 grades. Not only have these been developed, adapted and modified to the Liberian context, each of the eight Minimum Learning Competencies (MLCs) have been included in each textbook. So as to reach every high school student, for the first time in the country's history we have included the digitized form of the textbook accessible by a Quick Response (QR) code given in each book. Not only does it have the digitized textbook, but it provides additional learning materials for use by students, teachers and interested persons. The links to these e-resources and digitized material is being made available on the MoE's website.

The Textbooks and Teacher Guides have reached the hands of the students after a rigorous quality evaluation by carefully handpicked subject specialists by the MoE, to whom the Ministry expresses gratitude. For the success of this project, I acknowledge the contributions of the IRISE Project Team in the World Bank, and in particular, the Task-Team Leaders; the Project Implementation Team in Liberia headed by its Coordinator Abraham A. Kiazolu II, supported by the Executive Director of the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai and her technical working group (TWG), and the International Textbook Consultant and Advisor, Dr Shveta Uppal engaged by the MoE. These notwithstanding would not have been possible without the guidance of the Senior Management Team (SMT) of the Ministry of Education, and in particular, the Deputy Ministers for Instructions, Administration, and Planning, Research and Development, respectively.

Professor Dao Ansu Sonii, Sr.
Minister of Education
Republic of Liberia

Monrovia, Republic of Liberia
January 24, 2023

Acknowledgments

The development of textbooks contributes to the quality of teaching and learning that go on in the classroom.

The Ministry of Education (MoE) has aligned its Curriculum for Grades 10–12 to the National Curriculum Framework (NCF) of 2018. To ensure the provision of Teaching Learning Materials (TLMs) that support the revised curriculum, the Ministry has sought, reviewed and adapted a new set of textbooks and teacher guides along with digitized contents and e-learning resources for the five core subjects taught at the Senior Secondary education level, namely English Language and Literature, Mathematics, Biology, Chemistry and Physics, through an internationally competitive bidding process from the market supported by the World Bank funded Improving Results in Secondary Education (IRISE) Project.

With profound gratitude and honor, we recognize the Senior Management Team of the Ministry, headed by the Coach, Professor D. Ansu Sonii, Sr., for the strategic decision to make teaching learning materials available and accessible to all in the Liberian Senior Secondary School System, and for providing directions through the process of securing these textbooks and other teaching learning materials for our students and teachers. Our special thanks and appreciation to the World Bank for the financial support towards this policy intervention, and its education task-team including Alonso Sanchez, Oni Lusk-Stover and Binta B. Massaquoi for all their technical inputs offered throughout the process to ensure the kind of quality TLMs the Liberian students deserve are made available for improved learning outcomes.

We would like to specifically recognize the invaluable contributions of the 15 subject experts selected by the MoE from across the various education systems and the West African Examinations Council (WAEC) to evaluate, review and sign off on these teaching learning materials. They didn't just deliver according to our expectations, but also ensured the contextual relevance of the materials

to the Liberian Secondary Education Curriculum and its minimum learning competencies (MLCs). These subject experts include Professor Isaac Saye-Lakpoh Zawolo – *Superintendent* of the Monrovia Consolidated School System (MCSS), Mr. Matthew V.Z. Darblo, Sr. – *Mathematics Instructor* at the University of Liberia (UL), Mr. Charles Tieh Bropleh – *Mathematics Specialist* (MoE), Mrs. Linda Y. Dean – *English Specialist*, Mr. Hassan M. Bangura – *English Language and Literature Expert*, Mr. J. Emmanuel Milton – *English Specialist* (MoE), Mr. Moses K.M. Togbah – *Physics Specialist*, Mr. Prince A. Dossen – *Physics Specialist*, Mr. Benjamin Koryah – *Physics Instructor* at the University of Liberia (UL), Mr. Dominic Dugbe Doe – *Chemistry Specialist*, Mr. Patrick A. Anderson, Sr. – *Director o*f the Division of Technical and Vocational Education (MoE), Mr. Kandakai Massaquoi – *Chemistry Specialist*, Ms. Patricia N. Doe – *Head* of Biology Department, African Methodist Episcopal University (AMEU), Mr. Job Carpenter – *Biology Specialist* and Mr. Prince Philip K.A. Aderibigbe – *Biology Specialist*.

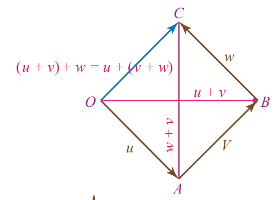
The MoE is sincerely grateful to Dr Shveta Uppal, the *International Textbook Consultant* engaged by the IRISE Project to provide technical guidance and quality assurance support to the revising of the Textbooks Management Guidelines (TMG) and the procurement process leading to the provision of textbooks, teacher guides, digital contents and e-learning resources for the Senior Secondary School System in Liberia in accordance with the revised TMG. Heartfelt thanks and appreciations also to the *Executive Director* for the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai, and members of her Technical Working Group (TWG) for taking up the responsibility to lead the process of making textbooks and other TLMs available to Liberian students and teachers.

Lastly, we acknowledge the IRISE Project Delivery Team led by Mr. Abraham A. Kiazolu, II – *Project Coordinator*, Mr. Fuseini A. Abu – *International Procurement Specialist* and Mr. Lawrence S. Taylor – *Project Control Specialist* who coordinated the entire process.

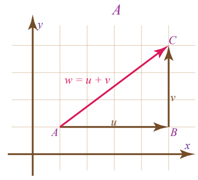
We remain grateful to you all!

Hon. Alexander N. Duopu, Sr.,
Deputy Minister for Instruction
Ministry of Education, Republic of Liberia
#The Teacher

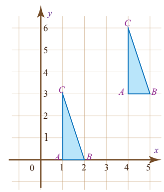
Contents

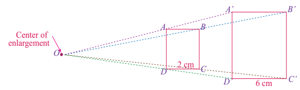


(u + v) + w = u + (v + w)
C
O
B
A
u + v
w
u

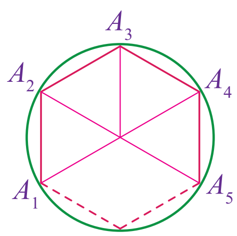


w = u + v
C
A
B

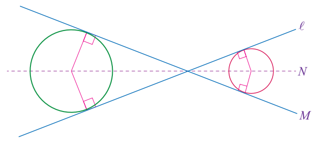




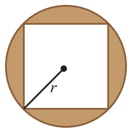
Center of enlargement
2 cm
6 cm



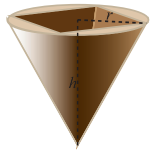
A_1
A_2
A_3
A_4
A_5



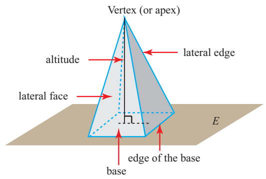
ℓ
N
M



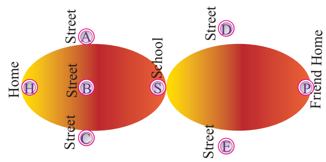
r



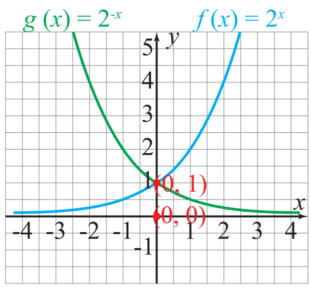
r
h



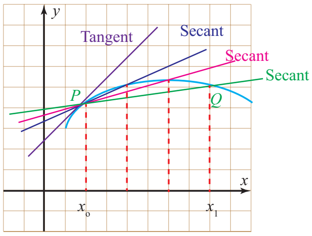
Vertex (or apex)
altitude
lateral edge
lateral face
E
edge of the base
base



Street
A
Home
H
Street
B
School
S
Street
C
Street
D
Friend Home
P
Street
E



$g(x) = 2^{-x}$
$f(x) = 2^{x}$
y
x
(0, 1)
(0, 0)


y
Tangent
Secant
Secant
Secant
P
Q
x
x_0
x_1

MATHEMATICS PART I

CONTENTS

GRADE 12

MATHEMATICS PART I

CONTENTS

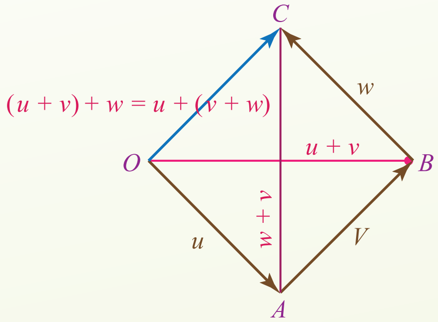
GRADE 12

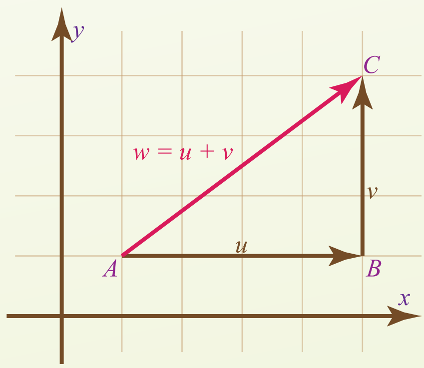
M12CH16

CHAPTER 16

VECTOR AND TRIGONOMETRY

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- describe vector using number ordered pair notation;
- describe the magnitude and direction of vectors;
- solve problems that involve vector addition and subtraction;
- multiply vectors by a number (scalar product);
- define and use unit vector and position vector;
- determine a vector from its direction and magnitude of vector;
- work with objects in static equilibrium;
- determine whether two vectors are parallel or orthogonal;
- determine Latitude and longitudes;
- calculate the lengths of objects of angles inscribed in a circle.

Introduction

The measurement of any physical quantity is always expressed in terms of a number and a unit. In physics, for example you come across a number of physical quantities like length, area, mass, volume, time, density, velocity, force, acceleration, and momentum, etc. Thus, most of the physical quantities can be divided into two categories as:

1. Physical quantities having magnitude only, called scalar quantities, and
2. Those having both magnitude and direction, called vector quantities.

Activity 1

1. Classify the measures of the following situations as scalar or vector.
   (a) The area of your classroom.
   (b) The flow of a river.
   (c) The number of students in your class room.
   (d) The distance and direction of your home from your school.
2. Classify each of the following quantities as either a quantity with only magnitude or a quantity with both magnitude and direction:
   (a) displacement
   (a) distance
   (b) speed
   (c) velocity
   (d) work
   (e) acceleration
   (f) area
   (g) time
   (h) weight
   (i) volume
   (j) density
   (k) force
   (l) momentum
   (m) temperature
   (n) mass

From you responses in Activity 1, observe that, quantities that are described by only their magnitude are called scalar quantities that are completely determined by the magnitude only and vector quantities are quantities that are determined using both magnitude and direction.

DEFINITION

1. A quantity which can be completely described by its magnitude expressed in some particular unit is called a scalar quantity.
2. A quantity which can be completely described by stating both its magnitude expressed in some particular unit and its direction is called a vector quantity.

EXAMPLE 1

Area, volume, mass, times, temperature, are all scalar quantities.

EXAMPLE 2

Force, velocity, acceleration are all vector quantities.

A vector is represented by an arrow with initial point and terminal point, the overhead arrow indicates the direction and the magnitude of the vector is the distance between the initial and terminal points of the arrow representing the vector.

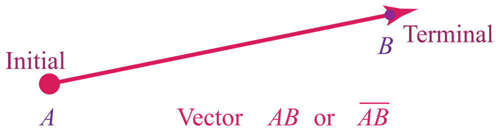


Figure 1.

DEFINITION

(Coordinate Form of a Vector in a Plane)

If $\boldsymbol{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (x, y), then the coordinate form of $\boldsymbol{v}$ is given by $\boldsymbol{v} = (x, y)$ or $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

The numbers x and y are called **components** or **coordinates** of $\boldsymbol{v}$.

Note:

1. If both the initial and terminal points lie at the origin, then $\boldsymbol{v}$ is the zero vector and is given by $\boldsymbol{v} = (0, 0)$ or $\mathrm{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
2. Two vectors $v = \begin{pmatrix} a \\ b \end{pmatrix}$ and $w = \begin{pmatrix} c \\ d \end{pmatrix}$ are said to be equal, denoted by $v = w$, if $a = c$ and $b = d$

3. The magnitude of vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$, denoted by $|v|$, is defined by $|v| = \sqrt{a^2 + b^2}$.

4. The direction of $v = \begin{pmatrix} a \\ b \end{pmatrix}$ is determined by the angle θ that v makes with the positive x-axis.

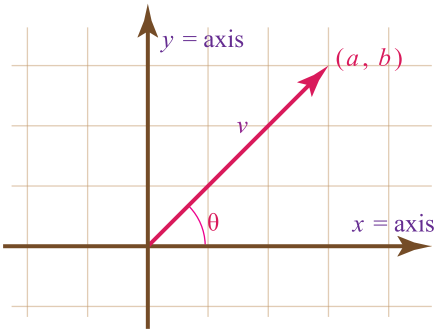


Figure 2.

The following procedure can be used to convert directed line segments to coordinate form and vice versa.

1. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the plane, then the coordinate form of the vector $\boldsymbol{v}$ represented by $\overrightarrow{PQ}$ is $\boldsymbol{v} = (x_2 - x_1, y_2 - y_1)$. Moreover, the magnitude of $\boldsymbol{v}$**,** denoted by $|v|$, is defined by:

$$|v| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. If $\boldsymbol{v} = (x, y)$, then $\boldsymbol{v}$ can be represented by the directed line segment in standard position, from O (0, 0) to $Q(x, y)$.

Example 3

Find the coordinate form and the magnitude of the vector v that has initial point $P(3, –7)$ and terminal point $Q(–2, 5)$.

Solution

Let $P(3, –7)$ and $Q(–2, 5)$. Then, the coordinate form of v is:

$$v = (-2 - 3, 5 - (-7)) = (-5, 12).$$

The magnitude of $\boldsymbol{v}$ is:

$$|v| = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Exercises

1. Fill in the blank spaces with the appropriate answer.
2. A directed line segment has a ______________ and a ______________. The magnitude of the directed line segment $\overrightarrow{PQ}$, denoted by ______________, is its ______________.
3. A vector whose initial point is at the origin (0, 0) can be uniquely represented by the coordinates of its terminal point (x, y). This is the ______________, written
4. $\boldsymbol{v} = (x, y)$, where x and y are the ______________ of $\boldsymbol{v}$.
5. The coordinate form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is $\overrightarrow{PQ}$ = ______________ $= \boldsymbol{v}$.
6. The magnitude (or length) of $\boldsymbol{v}$ is:
7. $|v| = \sqrt{\underline{\qquad\qquad}}$.
8. The coordinate form and magnitude of the vector $\boldsymbol{v}$ that has (1, 7) as its initial point and (4, 3) as its terminal point are ______________ and ______________.

Activity 2

Consider a displacement $\overrightarrow{AB}$ of 3 meters due north followed by a second displacement $\overrightarrow{BC}$ of 4 meters due east. Find the combined effect of these two displacements as a single displacement.

From Activity 2, you have seen that it is possible to add two vectors geometrically using tail to tip rule.

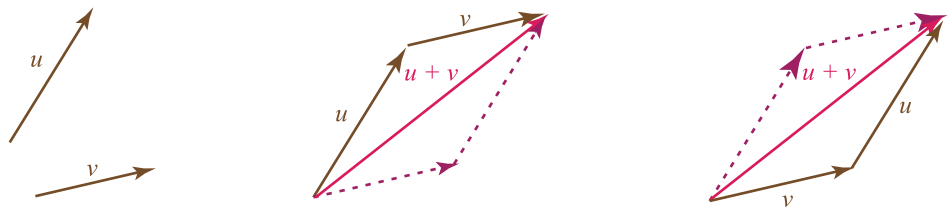


Figure 3.

To find $\boldsymbol{u} + \boldsymbol{v}$ *Move the initial point of* $\boldsymbol{v}$ *to the terminal point of* $\boldsymbol{u}$. **or** *Move the initial point of* $\boldsymbol{u}$ *to the terminal point of* $\boldsymbol{v}$.

Definition

(Addition of Vectors (Tail-to-Tip Rule))

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two vectors, the sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector determined as follows: Translate the vector $\boldsymbol{v}$ so that its initial point coincides with the terminal point of $\boldsymbol{u}$. The vector $\boldsymbol{u} + \boldsymbol{v}$ is represented by the arrow from the initial point of $\boldsymbol{u}$ to the terminal point of $\boldsymbol{v}$.

Theorem: (Commutative Property of Vector Addition)

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two vectors, then

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$$

Proof: Take any point O and draw the vectors $\overrightarrow{OA} = \boldsymbol{u}$ and $\overrightarrow{AB} = \boldsymbol{v}$ such that the terminal point of the vector $\boldsymbol{u}$ is the initial point of the vector $\boldsymbol{v}$ as show in Figure 4.

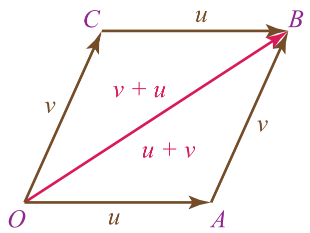


Figure 4.

Then, by definition of vector addition you have:

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{OB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, completing the parallelogram $OABC$ whose adjacent sides are OA and AB, you infer that $\overrightarrow{OC} = \overrightarrow{AB} = \boldsymbol{v}$, and $\overrightarrow{CB} = \overrightarrow{OA} = \boldsymbol{u}$. Using the triangle law of vector addition, you obtain

$$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\boldsymbol{v} + \boldsymbol{u} = \overrightarrow{OB} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we have:

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$$

Hence, vector addition is commutative. This is also called the parallelogram law of vectors.

Theorem: (Associative Property of Vector Addition)

If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are any three vectors, then

$$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}).$$

Proof: Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be three vectors represented by the line segments as shown in Figure 5. i.e., $\boldsymbol{u}=\overrightarrow{OA}$, $\boldsymbol{v}=\overrightarrow{AB}$, $\boldsymbol{w}=\overrightarrow{BC}$. Using the definition of vector addition, you have,

i.e., $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}$

$$\overrightarrow{OC}=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}\ \ldots\ldots\quad(1)$$

Again, you have,

$$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})$$

i.e. $\overrightarrow{OC}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$. . . . . (2)

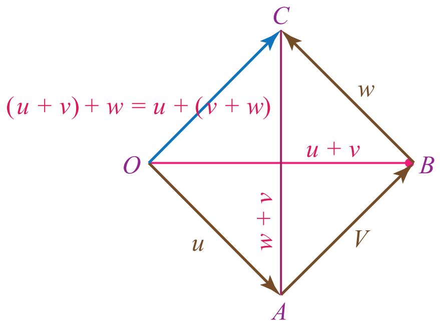


Figure 5.

Comparing 1 and 2, you have,

$$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$$

Hence, vector addition has associative property.

Exercises

1. A student walks a distance of 3 km due east, then another 4 km due south. Find the displacement relative to his starting point.
2. A car travels due east at 60 km/hr for 15 minutes, then turns and travels at 100 km/hr along a freeway heading due north for 15 minutes. Find the displacement from its starting point.
3. What is the resultant of the displacements 6 m north, 8 m east and 10 m North West?
4. If $ABCDEF$ is a regular polygon in which $\overrightarrow{AB}$ represents a vector $\boldsymbol{v}$ and $\overrightarrow{BC}$ represents a vector $\boldsymbol{w}$, express each of the vectors $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}$ and $\overrightarrow{AF}$ in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$.

DEFINITION

If $\boldsymbol{v}$ is a non-zero vector and k is a non-zero number (scalar), then the product $k\boldsymbol{v}$ is defined to be the vector whose length is $|k|$ times the length of $\boldsymbol{v}$ and whose direction is the same as that of $\boldsymbol{v}$ if $k > 0$ and opposite to that of $\boldsymbol{v}$ if $k < 0$.

$k\boldsymbol{v} = \mathbf{0}$ if $k = 0$ or $\boldsymbol{v} = \mathbf{0}$.

A vector of the form $k\boldsymbol{v}$ is called a scalar multiple of $\boldsymbol{v}$.

Theorem

Scalar multiplication satisfies the distributive laws, i.e. if k_1 and k_2 are any two scalars and $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors, then you have:

1. $(k_1 + k_2)\,\boldsymbol{u} = k_1\mathbf{u} + k_2\,\boldsymbol{u}$ 2. $k_1\,(\boldsymbol{u} + \boldsymbol{v}) = k_1\,\boldsymbol{u} + k_1\,\boldsymbol{v}$

Note

1. To obtain the difference $\boldsymbol{u} - \boldsymbol{v}$ without constructing $-\boldsymbol{v}$, position $\boldsymbol{u}$ and $\boldsymbol{v}$ so that their initial points coincide; the vector from the terminal point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{u}$ is then the vector $\boldsymbol{u} - \boldsymbol{v}$.
2. If $\boldsymbol{v}$ is any non-zero vector and $-\boldsymbol{v}$ is the negative of $\boldsymbol{v}$, then $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$.
3. For any three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, if $\boldsymbol{u} = \boldsymbol{v}$ and $\boldsymbol{v} = \boldsymbol{w}$, then $\boldsymbol{u} = \boldsymbol{w}$.
4. The zero vector $\mathbf{0}$ has the following property: For any vector $\boldsymbol{u}$, $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$.
5. For any vector $\boldsymbol{u}$, $1.\,\boldsymbol{u} = \boldsymbol{u}$.
6. If c and d are scalars and $\boldsymbol{u}$ is a vector, then $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.

The operations of vector addition and multiplication by a scalar are easy to work out in terms of coordinate forms of vectors. For the moment, we shall restrict the discussion to vectors in the plane.

If $\boldsymbol{u} = (x_1, y_1)$ and $\boldsymbol{v} = (x_2, y_2)$ are vectors and k is a real number, then

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2)\ ;\quad k\boldsymbol{u} = (kx_1, ky_1).$$

EXAMPLE 4

If $\boldsymbol{u} = (1, -2)$, $\boldsymbol{v} = (7, 6)$ and $k = 2$, find $\boldsymbol{u} + \boldsymbol{v}$ and $2\boldsymbol{u}$.

Solution

$\boldsymbol{u} + \boldsymbol{v} = (1 + 7, -2 + 6) = (8, 4)$, $2\boldsymbol{u} = (2(1), 2(-2)) = (2, -4)$

DEFINITION

If $\boldsymbol{u} = (x_1, y_1)$, $\boldsymbol{v} = (x_2, y_2)$, k is a scalar, then

$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2)$ and $k\boldsymbol{u} = (kx_1, ky_1)$.

EXAMPLE 5

If $\boldsymbol{u} = (1, -3)$ and $\boldsymbol{w} = (4, 2)$, then $\boldsymbol{u} + \boldsymbol{w} = (5, -1)$,
$2\boldsymbol{u} = (2, -6)$, $-\boldsymbol{w} = (-4, -2)$ and $\boldsymbol{u} - \boldsymbol{w} = (-3, -5)$

EXERCISES

1. Show that if $\boldsymbol{v}$ is a non-zero vector and m and n are scalars such that $m\boldsymbol{v} = n\boldsymbol{v}$, then $m = n$.
2. Let $\boldsymbol{u} = (1, 6)$ and $\boldsymbol{v} = (-4, 2)$. Find
   (a) $3\boldsymbol{u}$ (b) $3\boldsymbol{u} + 4\boldsymbol{v}$ (c) $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$
3. Draw diagrams to illustrate the following vector equations.
   (a) $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC}$ (b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AD}$

Given a vector $\boldsymbol{w}$, you may want to find two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are called components of $\boldsymbol{w}$ and the process of finding them is called resolving, or representing the vector into its vector components.

When you resolve a vector, you generally look for perpendicular components. Most often (in the plane), one component will be parallel to the x-axis and the other will be parallel to the y-axis. For this reason, they are often called the horizontal and vertical components of a vector.

In the Figure 6 below, the vector $\boldsymbol{w} = \overrightarrow{AC}$ is resolved as the sum of $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$.

The horizontal component of $\boldsymbol{w}$ is $\boldsymbol{u}$ and the vertical component is $\boldsymbol{v}$.

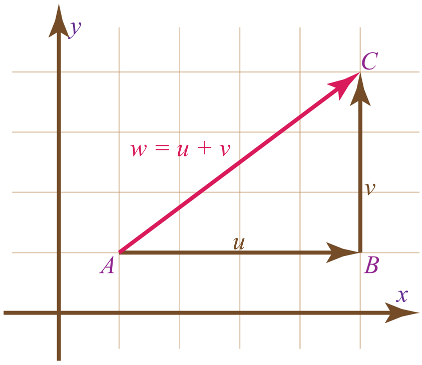


Figure 6.

EXAMPLE 6

A car weighting 8000 N is on a straight road that has a slope of 10^0 as shown in Figure 7. Find the force that keeps the car from rolling down.

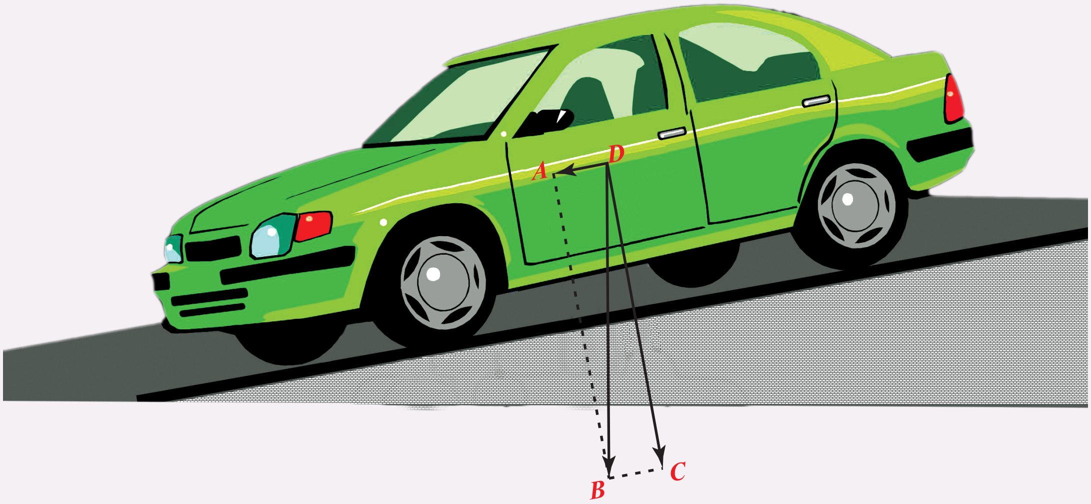


Figure 7.

Solution

The force vector $\overrightarrow{DB}$ acts in the downward direction.

this implies $\left|\overrightarrow{DB}\right| = 8000\,\text{N}$.

Observe that, $\overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{DB}$ and $m(\angle CDB) = 10^{\circ}$.

$\Rightarrow$ The force that keeps the car at D from rolling down is in the opposite direction of $\overrightarrow{DA}$

$$\Rightarrow \sin\ (\angle CDB) = \frac{\left|\overrightarrow{CB}\right|}{\left|\overrightarrow{DB}\right|} = \frac{\left|\overrightarrow{DA}\right|}{\left|\overrightarrow{DB}\right|} \quad \Rightarrow \sin 10^{\circ} = \frac{\overrightarrow{DA}}{8000\text{N}}$$

$$\Rightarrow \left|\overrightarrow{DA}\right| = 8000\text{ N} \times \sin 10^{\circ}\text{ N} = 1389.185\text{ N}.$$

Note

1. A given vector has an infinite number of pairs of possible component vectors. However, if directions of the component vectors are specified, the problem of resolving the vector into component vectors has a unique solution.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two non-zero vectors. In the expression $\boldsymbol{w} = k_1\boldsymbol{u} + k_2\boldsymbol{v}$

(a) the vectors $k_1\boldsymbol{u}$ and $k_2\boldsymbol{v}$ are said to be the components of $\boldsymbol{w}$ relative to $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

(b) the scalars k_1 and k_2 are called the coordinates of the vector $\boldsymbol{w}$ relative to $\boldsymbol{u}$ and $\boldsymbol{v}$.

DEFINITION

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be parallel (or collinear) if $\boldsymbol{u}$ and $\boldsymbol{v}$ lie either on parallel lines or on the same line.

Any vector whose magnitude is one is called a unit vector.

If $\boldsymbol{v}$ is any non-zero vector, the unit vector in the direction of $\boldsymbol{v}$ is obtained by multiplying vector $\boldsymbol{v}$ by $\frac{1}{|\boldsymbol{v}|}$. That is, the unit vector in the direction of $\boldsymbol{v}$ is $\frac{1}{|\boldsymbol{v}|}\cdot \boldsymbol{v}$

The unit vectors (1, 0) and (0, 1) are called the standard unit vectors in the plane.

Every pair of non-collinear vectors can be thought of as base. Of course, the components and the coordinates of a given vector in the plane will be different for different bases. For example, the vector $\boldsymbol{w}$ = (5, 8) can be written as

$$(5, 8) = (3, 2) + (2, 6) = (1, 6) + (4, 2) = (5, 0) + (0, 8), \text{ etc.}$$

Therefore, (3, 2) and (2, 6), (1, 6) and (4, 2), and (5, 0) and (0, 8), etc are components of $\boldsymbol{w}$.

Your main interest in this section is to find the horizontal and vertical components of a vector $\boldsymbol{w}$, denoted by w_x and w_y.

The unit vectors *i* and *j*

Vectors in the *xy* plane are represented based on the two special vectors $\boldsymbol{i}$ = (1, 0) and $\boldsymbol{j}$ = (0, 1). Notice that $|i| = |j| = 1$. i and j point in the positive directions of the x and y axes, respectively, as shown in Figure 8. These vectors are called standard unit base vectors.

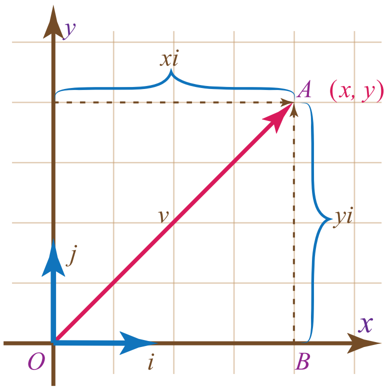


Figure 8.

Any vector $\boldsymbol{v}$ in the plane can be expressed uniquely in the form

$$\boldsymbol{v} = s\boldsymbol{i} + t\boldsymbol{j}$$

where s and t are scalars. In this case, you say that $\boldsymbol{v}$ is expressed as a linear combination of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Consider a vector $\boldsymbol{v}$ whose initial point is the origin and whose terminal point is the point $A = (x, y)$.

If $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = xi + yj.$$

If $\overrightarrow{PQ}$ is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2) as shown in Figure 9, then its position vector $\boldsymbol{v}$ is determined as

$$\boldsymbol{v} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\,\boldsymbol{i} + (y_2 - y_1)\,\boldsymbol{j}$$

Thus, $(x_2 - x_1)$ and $(y_2 - y_1)$ are the coordinates of $\boldsymbol{v}$ with respect to the base $\{\boldsymbol{i}, \boldsymbol{j}\}$.

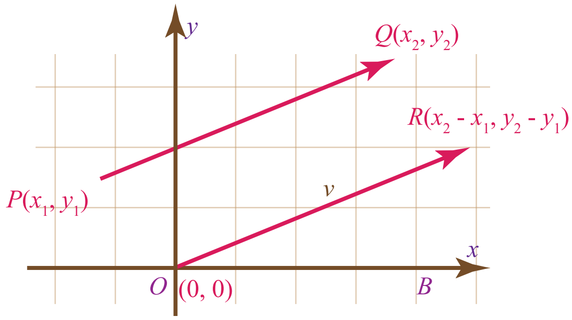


Figure 9.

Example 7

Express the following vectors in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ and find their norm.

(a) $v = (7, -8)$ (b) $u = (-1, 5)$ (c) $w = (-2, 3)$

Solution

(a) $v = (7, -8) = 7\boldsymbol{i} - 8\boldsymbol{j}$ and its norm (or magnitude) is

$$\sqrt{7^2 + (-8)^2} = \sqrt{49 + 64} = \sqrt{113}.$$

(b) $u = (-1, 5) = -1\boldsymbol{i} + 5\boldsymbol{j}$ and its norm (or magnitude) is $\sqrt{(-1)^2 + 5^2} = \sqrt{1 + 25} = \sqrt{26}.$

(c) $w = (-2, 3) = -2\boldsymbol{i} + 3\boldsymbol{j}$ with norm $\sqrt{13}$.

Example 8

Express each of the following as a vector in the coordinate form.

(a) $3\boldsymbol{i} + \boldsymbol{j}$ (b) $2\boldsymbol{i} - 2\boldsymbol{j}$ (c) $-\boldsymbol{i} + 6\boldsymbol{j}$

Solution

(a) $3\boldsymbol{i} + \boldsymbol{j} = 3(1, 0) + (0, 1) = (3, 0) + (0, 1) = (3, 1)$.

(b) $2\boldsymbol{i} - 2\boldsymbol{j} = 2(1, 0) - 2(0, 1) = (2, 0) + (0, -2) = (2, -2)$.

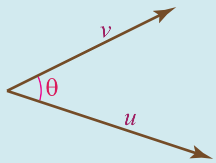
(c) $-\boldsymbol{i} + 6\boldsymbol{j} = -(1, 0) + 6(0, 1) = (-1, 0) + (0, 6) = (-1, 6)$.

Exercises

1. Find $\boldsymbol{u} + \boldsymbol{v}$ for each of the following pairs of vectors
   (a) $\boldsymbol{u} = (1, 4)$, $\boldsymbol{v} = (6, 2)$
   (b) $\boldsymbol{u} = (2, -2)$, $\boldsymbol{v} = (-2, 3)$
   (c) $\boldsymbol{u} = (7, -8)$, $\boldsymbol{v} = (-1, 6)$
   (d) $\boldsymbol{u} = (1 + \sqrt{2}, 0)$, $\boldsymbol{v} = (-\sqrt{2}, 2)$
2. Find the norm (or magnitude) of each of the following vectors.
   (a) $\boldsymbol{u} = (1, 1)$
   (b) $\boldsymbol{u} = (\frac{3}{2}, 0)$
   (c) $\boldsymbol{v} = (-2, 1)$
   (d) $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j}$, $x, y \in \mathbb{R}$
3. If $\boldsymbol{u} = 3\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$ and $\boldsymbol{v} = \frac{7}{2}\boldsymbol{i} - \frac{1}{4}\boldsymbol{j}$, find
   (a) $\boldsymbol{u} + \boldsymbol{v}$ (c) $t\boldsymbol{u}$, $t \in \mathbb{R}$
   (b) $\boldsymbol{u} - \boldsymbol{v}$ (d) $2\boldsymbol{u} - \boldsymbol{v}$
4. Answer each of the following
   (a) Find a unit vector in the direction of the vector (2, 4).
   (b) Find a unit vector in the direction opposite to the vector (1, 2).
   (c) Find two unit vectors, one in the same direction as, and the other opposite to the vector $\boldsymbol{u} = (x, y) \neq 0$.

Scalar (or Dot or) Product of Vectors

Definition



If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors and θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, then the dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$, is defined by:

$$\boldsymbol{u}\cdot\mathrm{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta.$$

Figure 10.

Example 9

Find the dot product of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ when

(a) $\boldsymbol{u} = (0,\ 1)$ and $\boldsymbol{v} = (0,\ 2)$

(b) $\boldsymbol{u} = (-2,\ 0)$ and $\boldsymbol{v} = (\sqrt{3},\ 3)$

Solution

Using the definition of dot product you have

(a) $|\boldsymbol{u}| = 1, |\boldsymbol{v}| = 2$ and $\theta = 0^\circ$ implies $\boldsymbol{u}\cdot\boldsymbol{v} = 1\times 2\times\cos 0^\circ = 2.$

(b) $|\boldsymbol{u}| = 2,\ |\boldsymbol{v}| = \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = 2\sqrt{3}$ and $\theta = 120^\circ$.

This implies $\boldsymbol{u}\cdot\boldsymbol{v} = 2\times 2\sqrt{3}\ \cos 120^\circ = -2\sqrt{3}.$

Note

- $\boldsymbol{i}\cdot\boldsymbol{j} = 0\ ,\ \boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = 1$
- If either $\boldsymbol{u}$ or $\boldsymbol{v}$ is $\mathbf{0}$, then $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.
- $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$ (dot product of vectors is commutative)
- If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then $\boldsymbol{u}\cdot\boldsymbol{v} = \pm|\boldsymbol{u}||\boldsymbol{v}|$. In particular, for any vector $\boldsymbol{u}$, we have $\boldsymbol{u}\cdot\boldsymbol{u} = |\boldsymbol{u}|^2$. Here, we write $\boldsymbol{u}^2$ to mean $|\boldsymbol{u}|^2$.
- If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, then $\boldsymbol{u}\cdot\boldsymbol{v} = 0$

  because $\cos\left(\frac{\pi}{2}\right) = 0$.

For purposes of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors.

In general, using the formula in the definition of the dot product, you can find the angle between two vectors. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, then the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by:

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}.$$

The following theorem lists the most important properties of the dot product. They are useful in calculations involving vectors.

Theorem

Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors and k be a scalar. Then,

(i) $k(\boldsymbol{u}\cdot\boldsymbol{v}) = (k\boldsymbol{u})\cdot\boldsymbol{v} = \boldsymbol{u}\cdot(k\boldsymbol{v})$ . . . . . . *(Associative Property of Dot Product)*

(ii) $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$ . . . . *Distributive Property of Dot Product)*

$\boldsymbol{u}\cdot\boldsymbol{u} > 0$ if $\boldsymbol{u} \neq \boldsymbol{0}$, and $\boldsymbol{u}\cdot\boldsymbol{u} = 0$ if $\boldsymbol{u} = \boldsymbol{0}$.

Corollary 1

If $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ are vectors, then $\boldsymbol{u}\cdot\boldsymbol{v} = u_1 v_1 + u_2 v_2$.

Proof: $\boldsymbol{u}\cdot\boldsymbol{v} = (u_1\boldsymbol{i}+u_2\boldsymbol{j})\cdot(v_1\boldsymbol{i}+v_2\boldsymbol{j})$

$= u_1\boldsymbol{i}\cdot(v_1\boldsymbol{i}+v_2\boldsymbol{j}) + u_2\boldsymbol{j}\cdot(v_1\boldsymbol{i}+v_2\boldsymbol{j})$

$= u_1\boldsymbol{i}\cdot v_1\boldsymbol{i} + u_1\boldsymbol{i}\cdot v_2\boldsymbol{j} + u_2\boldsymbol{j}\cdot v_1\boldsymbol{i} + v_2\boldsymbol{j}\cdot v_2\boldsymbol{j}$

$= u_1 v_1\ \boldsymbol{i}\cdot\boldsymbol{i} + u_1 v_2\cdot\boldsymbol{i}\cdot\boldsymbol{j} + u_2\boldsymbol{v}_1\boldsymbol{j}\cdot\boldsymbol{i} + u_2 v_2\ \boldsymbol{j}\cdot\boldsymbol{j}$

$= u_1 v_1 + u_2 v_2$. (Since $\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = 1$ and $\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{i} = 0$)

Example 10

Find the dot product of the vectors $\boldsymbol{u} = 3\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v} = 5\boldsymbol{i} - 3\boldsymbol{j}$

Solution

$\boldsymbol{u}\cdot\boldsymbol{v} = (3\boldsymbol{i}+2\boldsymbol{j})\cdot(5\boldsymbol{i}-3\boldsymbol{j}) = 3\times 5 + 2\times(-3) = 9.$

Application of the Dot Product of Vectors

The dot product has many applications. The following are examples of some of them.

Example 11

Find the angle between $3\boldsymbol{i} + 5\boldsymbol{j}$ and $-7\boldsymbol{i} + \boldsymbol{j}$.

Solution

Using vector method,

$$(3\boldsymbol{i} + 5\boldsymbol{j})\cdot(-7\boldsymbol{i} + \boldsymbol{j}) = 3(-7) + 5(1) = 16.$$

But by definition,

$$(3\boldsymbol{i} + 5\boldsymbol{j}) \cdot (7\boldsymbol{i} + \boldsymbol{j}) = |3\boldsymbol{i} + 5\boldsymbol{j}|\ |-7\boldsymbol{i} + \boldsymbol{j}|\ \cos\theta = \sqrt{9+25}\sqrt{49+1}\ \cos\theta$$

$$= \sqrt{34}\sqrt{50}\cos\theta = 16.$$

This implies $\cos\theta = \dfrac{16}{\sqrt{34}\sqrt{50}}$.

Therefore $\theta = \cos^{-1}\left(\dfrac{16}{\sqrt{34}\sqrt{50}}\right)$.

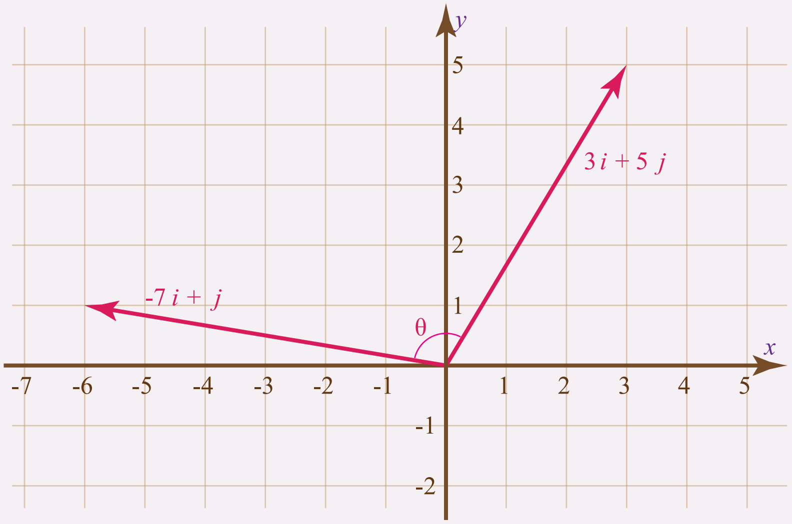


Figure 11.

The following are some other important properties of the dot product of vectors.

Corollary 2

$(\boldsymbol{u} - \boldsymbol{v})\cdot(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{u}^2 - \boldsymbol{v}^2$

$(\boldsymbol{u} \pm \boldsymbol{v})^2 = \boldsymbol{u}^2 \pm 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}^2$, where $\boldsymbol{u}^2 = \boldsymbol{u}\cdot\boldsymbol{u}$

EXAMPLE 12

Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors with $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\frac{\pi}{3}$.

(a) Evaluate $|3\boldsymbol{a} - 2\boldsymbol{b}|$

(b) Find the cosine of the angle between $3\boldsymbol{a} - 2\boldsymbol{b}$ and $\boldsymbol{a}$.

Solution

Using the properties of dot product we have,

(a) $|3\boldsymbol{a} - 2\boldsymbol{b}|^2 = (3\boldsymbol{a} - 2\boldsymbol{b})\cdot(3\boldsymbol{a} - 2\boldsymbol{b}) = 9|\boldsymbol{a}|^2 - 12\boldsymbol{a}\cdot\boldsymbol{b} + 4|\boldsymbol{b}|^2$

$= 9\times 16 - 12|\boldsymbol{a}||\boldsymbol{b}|\cos\frac{\pi}{3} + 4\times 49$

$= 144 - 12\times 4\times 7\times\frac{1}{2} + 196$

$= 172$

This implies $|3\boldsymbol{a} - 2\boldsymbol{b}| = \sqrt{172} = 2\sqrt{43}.$

(b) Let θ be the angle between $3\boldsymbol{a} - 2\boldsymbol{b}$ and $\boldsymbol{a}$. Then

$(3\boldsymbol{a} - 2\boldsymbol{b})\cdot\boldsymbol{a} = |3\boldsymbol{a} - 2\boldsymbol{b}||\boldsymbol{a}|\cos\theta \Rightarrow 3|\boldsymbol{a}|^2 - 2\boldsymbol{b}\cdot\boldsymbol{a} = 2\sqrt{43}\times 4\cos\theta$

Then $3\times 16 - 2|\boldsymbol{b}||\boldsymbol{a}|\cos\frac{\pi}{3} = 8\sqrt{43}\cos\theta$

This implies $48 - 2\times 7\times 4\times\frac{1}{2} = 8\sqrt{43}\cos\theta$

Therefore, $\cos\theta = \frac{2\sqrt{43}}{86}$.

The following statement shows how the dot product can be used to obtain information about the angle between two vectors.

Corollary 3

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be nonzero vectors. If θ is the angle between them, then

θ is acute, if and only if $\boldsymbol{u}\cdot\boldsymbol{v} > 0$

θ is obtuse, if and only if $\boldsymbol{u}\cdot\boldsymbol{v} < 0$

$\theta = \frac{\pi}{2}$ if and only if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$

Example 13

Determine the value of k so that the angle between the vectors

$\boldsymbol{u} = (k, 1)$ and $\boldsymbol{v} = (-2, 3)$ is

(a) acute

(b) obtuse

Solution

Using a direct application of Corollary 3, we have,

(a) $\boldsymbol{u}\cdot\boldsymbol{v} > 0 \Rightarrow (k, 1)\cdot(-2, 3) > 0 \Rightarrow -2k + 3 > 0 \Rightarrow k < \dfrac{3}{2}$

(b) $\boldsymbol{u}\cdot\boldsymbol{v} < 0 \Rightarrow k > \dfrac{3}{2}$.

Observe that the above vectors are perpendicular (orthogonal) if $k = \dfrac{3}{2}$.

Exercises

1. Find the vectors $\boldsymbol{z} = \boldsymbol{u} - 2(\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{z} = (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}$, where,
   (a) $\boldsymbol{u} = (8, 3)$, $\boldsymbol{v} = (-1, 2)$, $\boldsymbol{w} = (1, -4)$
   (b) $\boldsymbol{u} = \left(\dfrac{2}{3}, -\dfrac{1}{2}\right), \boldsymbol{v} = \left(-3.5, -\dfrac{4}{5}\right), \boldsymbol{w} = (-2, -1)$
2. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ make an angle $\theta = \dfrac{2}{3}\pi$. If $|\boldsymbol{u}| = 3$ and $|\boldsymbol{v}| = 4$, calculate
   (a) $\boldsymbol{u}\cdot\boldsymbol{v}$
   (b) $(\boldsymbol{u} - \boldsymbol{v})\cdot(\boldsymbol{u} + \boldsymbol{v})$
   (c) $(\boldsymbol{u} + \boldsymbol{v})\cdot(\boldsymbol{u} + \boldsymbol{v})$
   (d) $|2\boldsymbol{u} + \boldsymbol{v}|$
3. Using properties of the scalar product, show that for any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{z}$,
   (a) $(\boldsymbol{u} + \boldsymbol{v})^2 = \boldsymbol{u}^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}^2$
   (b) $(\boldsymbol{u} - \boldsymbol{v})^2 = \boldsymbol{u}^2 - 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}^2$
   (c) $(\boldsymbol{u} + \boldsymbol{v})\cdot(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{u}^2 - \boldsymbol{v}^2$
   (d) $(\boldsymbol{u} + \boldsymbol{v})\cdot(\boldsymbol{w} + \boldsymbol{z}) = \boldsymbol{u}\cdot\boldsymbol{w} + \boldsymbol{u}\cdot\boldsymbol{z} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{v}\cdot\boldsymbol{z}$
4. Let $\boldsymbol{u} = (1, -1)$, $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (-2, 3)$. Find the cosines of the angles between
   (a) $\boldsymbol{u}$ and $\boldsymbol{v}$
   (b) $\boldsymbol{v}$ and $\boldsymbol{w}$
   (c) $\boldsymbol{u}$ and $\boldsymbol{w}$
5. Prove that if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$ for all non-zero vectors $\boldsymbol{v}$, then $\boldsymbol{u} = \boldsymbol{0}$.
6. Show that $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$ are perpendicular to each other if and only if $|\boldsymbol{u}| = |\boldsymbol{v}|$

7. Show that $(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v}) \geq (\boldsymbol{u}\cdot\boldsymbol{v})^2$. When is equality holds true?
8. Show that $\boldsymbol{u}\cdot\boldsymbol{v} = 0 \Leftrightarrow |\boldsymbol{u}+\boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2$.
9. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ make an angle $\theta = \frac{\pi}{6}$. If $|\boldsymbol{u}| = \sqrt{3}$ and $|\boldsymbol{v}| = 1$, then find
   (a) $|\boldsymbol{u}+\boldsymbol{v}|$ (b) $|\boldsymbol{u}-\boldsymbol{v}|$
10. Let $|\boldsymbol{u}| = 13$, $|\boldsymbol{v}| = 19$ and $|\boldsymbol{u}+\boldsymbol{v}| = 24$. Calculate
   (a) $\boldsymbol{u}\cdot\boldsymbol{v}$ (b) $|\boldsymbol{u}-\boldsymbol{v}|$ (c) $|3\boldsymbol{u}+4\boldsymbol{v}|$

Applications of Vectors

From previous knowledge, you notice that vectors have many applications. Geometrically, any two points in the plane determine a straight line. Also a straight line in the plane is completely determined if its slope and if a point through which it passes are known. These lines have been determined to have a certain direction. Thus, related to vectors, you will see how one can write equations of lines and circles using vectors.

Example 14

Show that, in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Solution

Let ΔABC be a given right-angled triangle with $\angle C = 90^{\circ}$.

Consider the vectors, $\overrightarrow{AC}, \overrightarrow{CB}$ and $\overrightarrow{AB}$ as shown in Figure 12.

Since $\angle C = 90^{\circ}, \overrightarrow{AC}.\overrightarrow{CB} = 0$. By sector addition you have $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$. Thus

$$\overrightarrow{AB}^2 = \overrightarrow{AB}.\overrightarrow{AB} = (\overrightarrow{AC}+\overrightarrow{CB}).(\overrightarrow{AC}+\overrightarrow{CB}) = \overrightarrow{AC}^2 + \overrightarrow{CB}.\overrightarrow{AC} + \overrightarrow{CB}^2$$

$$= \overrightarrow{AC}^2 + \overrightarrow{CB}^2 \ldots\ldots\ldots\ldots \text{ since } \overrightarrow{CB}.\overrightarrow{AC} = 0$$

Hence $\overrightarrow{AB}^2 = \overrightarrow{AC}^2 + \overrightarrow{CB}^2$.

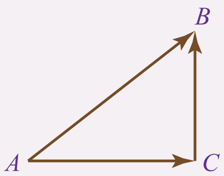


Figure 12.

Example 15

Show that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent (i.e. they intersect at a single point).

Solution

Let ABC be a given triangle and AD and BE be perpendiculars on BC and CA respectively. Suppose AD and BE meet at O as shown in Figure 13.

Consider the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ and $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$.

Observe that $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

According to our hypothesis, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are perpendicular. Thus

$$\overrightarrow{BC}.\overrightarrow{AD} = 0$$

$$\Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{AD} = 0 \quad \Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{OA} = 0$$

$$\Rightarrow \overrightarrow{OC}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OA} \quad \ldots\ldots\ldots\ldots (1)$$

Similarly, we can write for $\overrightarrow{BE}$ and $\overrightarrow{CA}$, i.e., $\overrightarrow{BE}.\overrightarrow{CA} = 0$

$$\Rightarrow \overrightarrow{BE}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0 \Rightarrow \overrightarrow{OB}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0$$

$$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC} \quad \ldots\ldots\ldots\ldots (2)$$

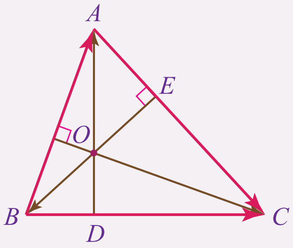


Figure 13.

By adding 1 and 2, we obtain

$$\overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OB}.\overrightarrow{OC} \Rightarrow \overrightarrow{OC}.(\overrightarrow{OB} - \overrightarrow{OA}) = 0 \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0.$$

Hence $\overrightarrow{BA}$ and $\overrightarrow{CF}$ are perpendicular.

Thus, the perpendiculars from A, B and C to the opposite sides are concurrent.

Example 16

Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

Solution

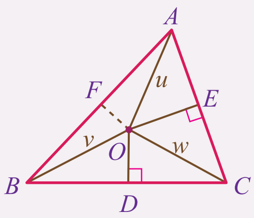


Figure 14.

Let ABC be a triangle and D, E, F the mid-points of BC, CA, and AB, respectively.

DO and EO are perpendiculars to BC and CA respectively. Join O to the mid-point F of AB.

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

Then, $\overrightarrow{BC} = \boldsymbol{w} - \boldsymbol{v}$ and $\quad \overrightarrow{OD} = \dfrac{\boldsymbol{v} + \boldsymbol{w}}{2}$

Since $\overrightarrow{OD}$ and $\overrightarrow{BC}$ are perpendicular, you have

$$\overrightarrow{OD}.\overrightarrow{BC} = 0. \text{ i.e. } \left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2}\right).(\boldsymbol{w}-\boldsymbol{v}) = 0 \quad \ldots\ldots\ldots \textbf{(1)}$$

Similarly, since $\overrightarrow{OE}$ and $\overrightarrow{CA}$ are perpendicular, you get

$$\left(\frac{\boldsymbol{w}+\boldsymbol{u}}{2}\right).(\boldsymbol{u}-\boldsymbol{w}) = 0 \quad \ldots\ldots\ldots\ldots \textbf{(2)}$$

From (1) and (2), you obtain $\boldsymbol{u}^2 - \boldsymbol{v}^2 = 0$ or $\boldsymbol{v}^2 - \boldsymbol{u}^2 = 0$

$$\Rightarrow \frac{1}{2}(\boldsymbol{v}+\boldsymbol{u}).(\boldsymbol{v}-\boldsymbol{u}) = 0 \Rightarrow \overrightarrow{OF} \text{ and } \overrightarrow{BA} \text{ are perpendicular.}$$

Apart from the applications discussed above, vectors have many practical applications. Some are presented in the following subunits.

Practical application of vectors

Previously, you saw how vectors are useful in determining the equations of a line, and the equations of a tangent line to a circle. Now, you will consider practical problems and applications involving vectors.

Example 17

Show that the vectors $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{v} = (0.5, 1)$ are two parallel vectors which are of the same direction whereas the vectors $\boldsymbol{u}_1 = (-1, 2)$ and $\boldsymbol{v}_1 = (0.5, -1)$ are in opposite directions.

Solution

Consider $\boldsymbol{u}.\boldsymbol{v}$ and $\boldsymbol{u}_1.\boldsymbol{v}_1$.

$\boldsymbol{u}.\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta \Rightarrow \frac{5}{2} = \sqrt{5}\times\frac{\sqrt{5}}{2}.\cos\theta \Rightarrow \cos\theta = 1$ and hence $\theta = 0$.

Thus, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and have the same direction.

Similarly, $\boldsymbol{u}_1.\boldsymbol{v}_1 = |\boldsymbol{u}_1||\boldsymbol{v}_1|\cos\theta \Rightarrow -\frac{5}{2} = \sqrt{5}\times\frac{\sqrt{5}}{2}\cos\theta$

$\Rightarrow \cos\theta = -1$ and hence $\theta = \pi$.

Therefore, $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ are parallel and have opposite direction.

Example 18

If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{z}$ are vectors from the origin to the points A, B, C and D, respectively, and $\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{w} - \boldsymbol{z}$, prove that $ABCD$ is a parallelogram.

Solution

Let O be the fixed origin of these vectors.

Since $\boldsymbol{v} - \boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{w} - \boldsymbol{z} = \overrightarrow{DC}$, you have $\overrightarrow{AB} = \overrightarrow{DC}$.

$\Rightarrow$ The vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel and equal.

Also, $\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{w} - \boldsymbol{z} \Rightarrow \boldsymbol{w} - \boldsymbol{v} = \boldsymbol{z} - \boldsymbol{u} \Rightarrow \overrightarrow{BC} = \overrightarrow{AD}$.

Thus, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel and equal. Hence, $ABCD$ is a parallelogram.

Example 19

Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution

Let ABC be a triangle and D, E, F the mid-points of the sides BC, CA, and AB, respectively, as shown in Figure 15.

First, consider the triangle ABD. You have

$$\overrightarrow{AD} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \quad \ldots\ldots\ldots\ldots\ (1)$$

In the same way, you see that

$$\overrightarrow{BE} = \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA} \quad \ldots\ldots\ldots\ldots\ (2)$$

and $$\overrightarrow{CF} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} \quad \ldots\ldots\ldots\ (3)$$

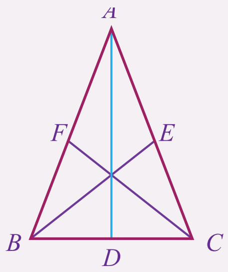


Figure 15.

Adding up 1, 2 and 3, you get

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = \frac{3}{2}.0 = 0.$$

Example 20

A video camera weighing 15 pounds is going to be suspended by two wires from the ceiling of a room as shown in Figure 16. What is the resulting tension in each wire?

Solution

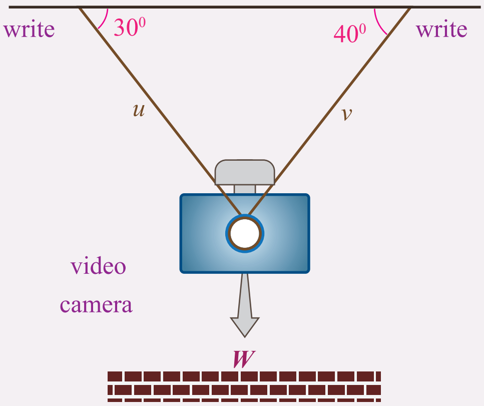


Figure 16.

The force vector of the camera is straight down, so $\boldsymbol{w} = (0, -15)$.

Vector $\boldsymbol{u}$ has magnitude $|\boldsymbol{u}|$ and can be represented as

$(-|\boldsymbol{u}|\cos 30^\circ, |\boldsymbol{u}|\sin(30^\circ))$.

Similarly, $\boldsymbol{v} = (|\boldsymbol{v}|\cos 40^\circ, |\boldsymbol{v}|\sin 40^\circ)$.

Since the system is in equilibrium, the sum of the force vectors is $\mathbf{0}$.

$\Rightarrow \mathbf{0} = \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

$= (-|\boldsymbol{u}|\cos 30^\circ + |\boldsymbol{v}|\cos 40^\circ + 0,$
$|\boldsymbol{u}|\sin 30^\circ + |\boldsymbol{v}|\sin 40^\circ - 15)$

From the components of the vector equation, you have two equations,

$0 = -|\boldsymbol{u}|\cos 30^\circ + |\boldsymbol{v}|\cos 40^\circ$

$0 = |\boldsymbol{u}|\sin 30^\circ + |\boldsymbol{v}|\sin 40^\circ - 15$

that you want to solve for the tensions $|\boldsymbol{u}|$ and $|\boldsymbol{v}|$.

From the first, you get $|\boldsymbol{u}|\cos 30^\circ = |\boldsymbol{v}|\cos 40^\circ \Rightarrow |\boldsymbol{v}| = |\boldsymbol{u}|\dfrac{\cos 30^\circ}{\cos 40^\circ}$.

Substituting this value for $|\boldsymbol{v}|$ into the second equation you have

$$0 = |\boldsymbol{u}|\sin 30^\circ + |\boldsymbol{u}|\frac{\cos 30^\circ}{\cos 40^\circ}.\sin 40^\circ - 15$$

$$\Rightarrow |\boldsymbol{u}| = \frac{15}{\sin 30^\circ + (\cos 30^\circ)(\tan 40^\circ)} \cong 12.2 \text{ pounds}$$

Putting this value back into

$|\boldsymbol{v}| = |\boldsymbol{u}|\dfrac{\cos 30^\circ}{\cos 40^\circ}$, you get $|\boldsymbol{v}| = (12.2)\dfrac{\cos(30^\circ)}{\cos(40^\circ)} \cong 13.9$ pounds.

Exercises

1. Are the points A, B and C collinear?
   (a) $A(1, -4)$, $B(-2, -3)$, $C(11, -11)$
   (b) $A(-2, -3)$, $B(4, 9)$, $C(-11, -21)$
2. If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{z}$ are vectors from the origin to the points A, B, C, D, respectively, and $\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{w} - \boldsymbol{z}$, then show that $ABCD$ is a parallelogram.

3. Figure 17 shows the magnitudes and directions of six coplanar forces (forces on the same plane).

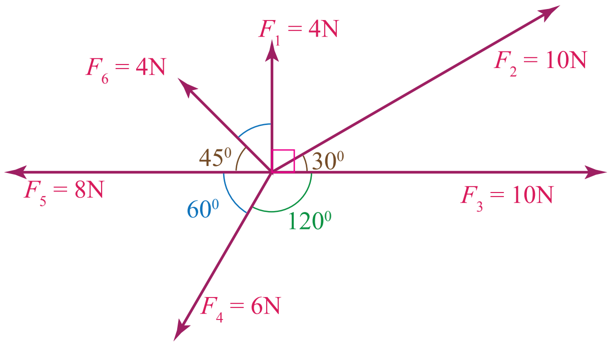


Figure 17.

Find each of the following dot products.

(a) $F_1 . F_2$

(b) $F_5 . F_6$

(c) $(F_1 + F_2 - F_3) . (F_4 + F_5 - F_6)$

4. Let $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} + 3\boldsymbol{j}$ be vectors. Find the unit vectors in the direction of each of the following vectors.

(a) $a + b$

(b) $2a + b - \frac{3}{2}c.$

5. Three forces $\boldsymbol{F}_1 = 2\boldsymbol{i} + 3\boldsymbol{j}$, $\boldsymbol{F}_2 = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{F}_3 = 3\boldsymbol{i} - \boldsymbol{j}$ measured in Newton act on a particle causing it to move from $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j}$ to $\boldsymbol{B} = 3\boldsymbol{i} + 4\boldsymbol{j}$ where AB is measured in meters. Find the total work done by the combined forces.

Key Terms

- Coordinate form of a vector
- Initial point
- Parallel vectors
- Perpendicular (orthogonal) vectors
- Resolution of vectors
- Rigid motion
- Scalar quantity
- Standard unit vector
- Standard position
- Terminal point
- Unit vector
- Vector quantity
- Zero vector

Summary

- **Vector**
  (i) A quantity which can be completely described by its magnitude expressed in some particular unit is called a scalar quantity.
  (ii) A quantity which can be completely described by stating both its magnitude expressed in some particular unit and its direction is called a vector quantity.
  (iii) Two vectors are said to be equal if they have the same magnitude and direction.
  (iv) A zero vector or null vector is a vector whose magnitude is zero and whose direction is indeterminate.
  (v) A unit vector is a vector whose magnitude is one.

- **Addition of vectors**
  Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors, then the sum $\boldsymbol{u}+\boldsymbol{v}$ is a vector given by the parallelogram law or triangle law satisfying the following properties.
  (i) Vector addition is commutative. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
  (ii) Vector addition is associative. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$
  (iii) $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$
  (iv) $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$
  (v) $|\boldsymbol{u}+\boldsymbol{v}|\leq|\boldsymbol{u}|+|\boldsymbol{v}|$

- **Multiplication of a vector by a scalar**
  Let $\boldsymbol{u}$ be a vector and λ be a scalar, then $\lambda\boldsymbol{u}$ is a vector satisfying the following properties.
  (i) $|\lambda\boldsymbol{u}|=|\lambda||\boldsymbol{u}|$
  (ii) If μ is a scalar, then $(\lambda+\mu)\boldsymbol{u}=\lambda\boldsymbol{u}+\mu\boldsymbol{u}$
  (iii) If $\boldsymbol{v}$ is a vector, then $\lambda(\boldsymbol{u}+v)=\lambda\boldsymbol{u}+\lambda\boldsymbol{v}$.

- **Scalar product or dot product**
  The dot product of two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$ and θ is an angle between them is defined as: $\boldsymbol{u}.\boldsymbol{v}=|\boldsymbol{u}||\boldsymbol{v}|\cos\theta$ satisfying the following properties.
  (i) The scalar product of vectors is commutative. $\boldsymbol{u}.\boldsymbol{v}=\boldsymbol{v}.\boldsymbol{u}$.
  (ii) If $\boldsymbol{u}=\mathbf{0}$ or $\boldsymbol{v}=\mathbf{0}$, then $\boldsymbol{u}.\boldsymbol{v}=0$.
  (iii) Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u}.\boldsymbol{v}=0$.

Exercises

1. Given vectors $\boldsymbol{u} = (2, 5)$, $\boldsymbol{v} = (-3, 3)$ and $\boldsymbol{w} = (5, 3)$
   (a) Find $\boldsymbol{u} - \boldsymbol{v} + 2\boldsymbol{w}$ and $|\boldsymbol{u} - \boldsymbol{v} + 2\boldsymbol{w}|$
   (b) Find $2\boldsymbol{u} + 3\boldsymbol{v} - \boldsymbol{w}$ and $|2\boldsymbol{u} + 3\boldsymbol{v} - \boldsymbol{w}|$
   (c) Find the unit vector in the direction of $\boldsymbol{u}$
   (d) Find $\boldsymbol{z}$ if $\boldsymbol{z} + \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$
   (e) Find $\boldsymbol{z}$ if $\boldsymbol{u} + 2\boldsymbol{z} = 3\boldsymbol{v}$
2. When two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ with $|\boldsymbol{F}_1| = 30$N and $|\boldsymbol{F}_2| = 40$N act on a point, if the angle between $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is 30^o, then find the magnitude of the resultant force.
3. A rotation R takes A (1, –3) to A' (3, 5) and B (0, 0) to B' (4, –6). Find the centre of rotation.
4. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors with $|\boldsymbol{a}| = |\boldsymbol{b}|$, show that $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$ are orthogonal.
5. A person pulls a body 50 m on a horizontal ground by a rope inclined at 30^o to the ground. Find the work done by the horizontal component of the tension in the rope if the magnitude of the tension is 10N.
6. Prove that the sum of all vectors from the centre of a regular polygon to each side is $\mathbf{0}$.
7. Using a vector method, prove that an angle inscribed in a semi-circle measures 90^o.
8. Find the resultant of two vectors of magnitudes 6 units and 10 units, if the angle between them is:
   (a) 30^o
   (b) 120^o
   (c) 150^o
9. Four forces acting on a particle are represented by $3\boldsymbol{i} + 4\boldsymbol{j}$, $3\boldsymbol{i} - 5\boldsymbol{j}$, $5\boldsymbol{i} + 4\boldsymbol{j}$ and $2\boldsymbol{i} + \boldsymbol{j}$. Find the resultant force $\boldsymbol{F}$.

10. Three towns A, B and C are joined by straight railways. Town B is 600 km east and 1200 km north of town A. Town C is 800 km east and 900 km south of town B. By considering town A as the origin,

    (a) Find the position vectors of B and C using the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

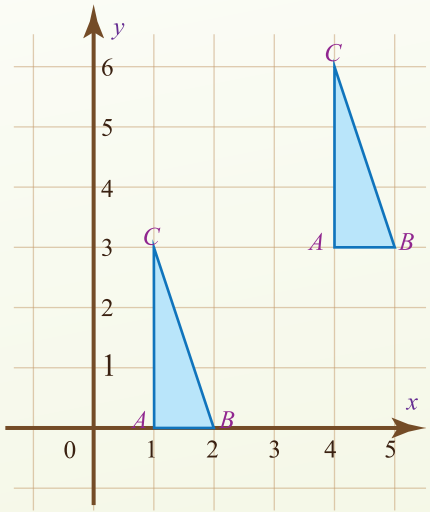
    (b) If T is a train station two thirds of the way along the rail way form town A to town B, prove that T is the closest station to town C on the rail way from town A to town B.

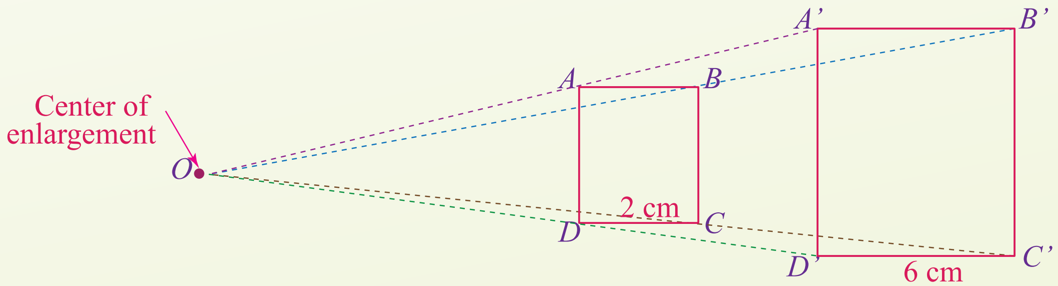
CHAPTER 17

M12CH17

TRANSFORMATIONS

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- define and discuss transformation using concept of movement or congruency;
- find the coordinates of transformations;
- define, discuss and apply the concept of reflection;
- define, discuss and apply the concept of similarities;
- define, discuss and apply the concept of translation.

Introduction

Transformations are of practical importance, especially in solving problems and describing difficulties in simpler forms. Transformations can be managed in different forms, those that maintain direction and those that change direction. There are many versions of transformations and two types of transformations namely translations and reflections will be considered.

Activity 1

1. In which of the following conditions does the shape or size or both of the object change.
   (a) When a rubber is stretched.
   (b) When a commercial jet flies from place to place at a specific time.
   (c) When the earth rotates about its axis.
   (d) When you see your image in a plane mirror.
   (e) When you draw the map of your school compound.
2. Let T be a mapping of the plane onto itself given by $T((x, y)) = (x + 1, -y)$.
   For example, $T((4, 3)) = (4 + 1, -3) = (5, -3)$.
   If $A = (0, 1)$, $B = (-3, 2)$ and $C = (2, 0)$, find the coordinates of the image of A, B and C.
   Find the image of ΔABC under T. Is ΔABC congruent to its image?
3. Suppose T is a mapping of the plane onto itself which sends point P to point P'.
   Let $A = (2, -3)$ and $B = (5, 4)$. Compare the lengths of AB and $A'B'$ when
   (a) $T((x, y)) = (x, 0)$
   (b) $T((x, y)) = (x, -y)$
   (c) $T((x, y)) = (x + 1, y - 3)$
   (d) $T((x, y)) = \left(\frac{1}{2}x, 2y\right)$

From your responses of Activity 1, you have seen that some mappings called Transformations of the plane onto itself preserve shape, size or distance between any two points. Based on this, transformations are classified as either rigid motion or non-rigid motion.

Definition

Rigid motion

A motion is said to be rigid motion, if it preserves distance. That is for $P \neq Q$, $PQ = P'Q'$ where P' and Q' are the images of P and Q, respectively. Otherwise it is said to be non-rigid motion.

A transformation is said to be an identity transformation, if the image of every point is itself. For example, if an object is rotated 360° it is an identity transformation.

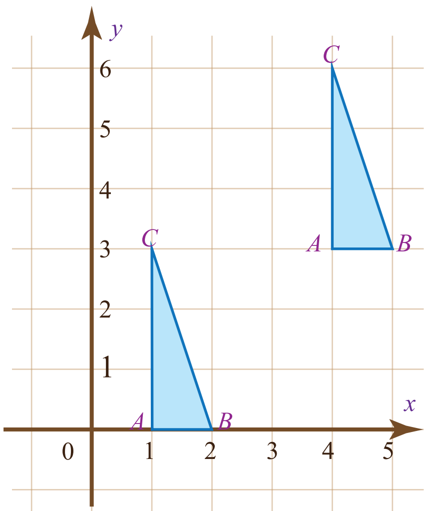
Note: Rigid motion carries any plane figure to a congruent plane figure, i.e., it carries triangles to congruent triangles, rectangles to congruent rectangles, etc.

An identity transformation is a rigid motion.

In this unit two different types of rigid motions are presented.

Figure 1.

Consider the following coordinate plane.



When ΔABC is transformed to $\Delta A'B'C'$, AB and $A'B'$ are parallel to the x-axis, and AC and $A'C'$ are parallel to the y-axis. Moreover, ΔABC and $\Delta A'B'C'$ have the same orientation. i.e., the way they face is the same. This type of transformation is said to be a translation.

Definition

In a transformation, if every point of a figure is moved along the same direction through the same distance, then the transformation is called a translation or parallel movement.

If point P is translated to point P', then the vector $\overrightarrow{PP'}$ is said to be the translation vector.

If $\boldsymbol{u} = (h, k)$ is a translation vector, then the image of the point (x, y) under the translation will be the point $(x + h, y + k)$.

Example 1

Let T be a translation that takes the origin to (1, 2). Determine the translation vector and find the images of the following points.

(a) (2,–1) (b) (–3, 5) (c) (1, 2)

Solution

$T((0, 0)) = (1, 2)$ implies $\boldsymbol{u} = (1, 2)$ is the translation vector.

This implies, for any (x, y) in the coordinate plane $T(x, y)$ $(x + 1, y + 2)$.

Thus,

(a) $T((2, -1)) = (2 + 1, -1 + 2) = (3, 1)$.

(b) $T((-3, 5)) = (-3 + 1, 5 + 2) = (-2, 7)$.

(c) $T((1, 2)) = (1 + 1, 2 + 2) = (2, 4)$.

Example 2

Let the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be translated by the vector $\boldsymbol{u} = (h, k)$ to P' and Q' respectively. Show that $|\overrightarrow{PQ}| = |\overrightarrow{P'Q'}|$.

Solution

The translations T of any point (x, y) is given by $T(x, y) = (x + h, y + k)$.

After the translation $P'(x_1 + h, y_1 + k)$ and $Q'(x_2 + h, y_2 + k)$.

Then, $$|\overrightarrow{P'Q'}| = \sqrt{(x_2 + h - x_1 - h)^2 + (y_2 + k - y_1 - k)^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |\overrightarrow{PQ}|.$$

The above example shows that a translation is a rigid motion, a motion that does not change shape and size of figures.

Translation formula in terms of coordinates of the plane.

1. If (h, k) is a translation vector, then

   (a) the origin is translated to (h, k) i.e., $(0, 0) \rightarrow (h, k)$

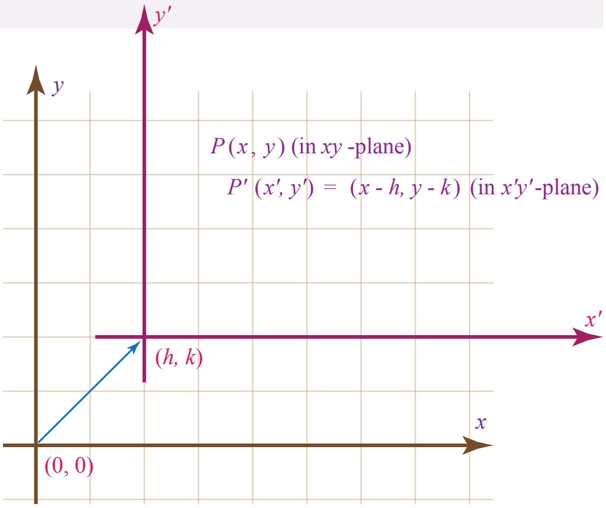


Figure 2.

(b) the point $P(x, y)$ is translated to $P'(x + h, y + k)$.

2. If the translation vector is $\overline{AB}$ where $A = (a, b)$ and $B = (c, d)$, then
   (a) the origin is translated to $(c - a, d - b)$, and
   (b) the point $P(x, y)$ is translated to $(x + c - a, y + d - b)$

Example 3

If a translation T takes the origin to $P'(1, 2)$, then the translation vector is $(1, 2)$ and for any (x, y) in the coordinate plane, $T(x, y) = (x + 1, y + 2)$.

In particular, $T(-2, 3) = (-2 + 1, 3 + 2) = (-1, 5)$.

Example 4

If a translation T takes the origin to $(-1, 1)$, then find

(a) the images of the points $P(1, 3)$ and $Q(-3, 6)$.

(b) the image of the triangle with vertices $A(2, -2)$, $B(-3, 2)$ and $C(4, 1)$.

Solution

The translation vector is $(-1, 1)$ and for any (x, y) in the coordinate plane, $T(x, y) = (x - 1, y + 1)$.

(a) The image of the point $P(1, 3)$ is $T(1, 3) = (1 + (-1), 3 + 1) = (0, 4)$.
The image of the point $Q(-3, 6)$ is $T(-3, 6) = (-3 -1, 6 +1) = (-4, 7)$.

(b) The image of the triangle can be obtained by transforming only the vertices of the triangle.
$T(2, -2) = (2 + (-1), -2 + 1) = (1, -1)$, $T(-3, 2) = (-3 + (-1), 2 + 1) = (-4, 3)$ and
$T(4, 1) = (4 + (-1), 1 + 1) = (3, 2)$.
Thus, the images of A, B and C are $A'(1, -1)$, $B'(-4, 3)$ and $C'(3, 2)$ respectively.
That is, the image ΔABC is $\Delta A'B'C'$.

Example 5

If a translation T takes the point $(-1, 3)$ to the point $(4, 2)$, then find the images of the following lines under the translation T.

(a) $\ell : y = 2x - 3$

(b) $\ell : 5y + x = 1$

Solution

The translation vector is $(h, k) = (4 - (-1), 2 - 3) = (5, -1)$. Thus, for any point (x, y) in the coordinate plane $T(x + 5, y - 1)$.

A translation maps lines onto parallel lines. Let ℓ' be the image of ℓ under T. Then,

(a) ℓ': $T(y) = 2T(x) - 3$.

Thus, the image of ℓ: $y = 2x - 3$ is ℓ': $y = 2x + 8$.

(b) The image of ℓ : $5y + x = 1$ is ℓ': $5(y + 1) + (x - 5) = 1$.

This implies ℓ': $5y + x = 1$.

Exercises

1. If a translation T takes the origin to the point (–3, 2), find the image of the rectangle $ABCD$ with vertices $A(3, 1)$, $B(5, 1)$, $C(5, 4)$ and $D(3, 4)$.
2. Triangle ABC is transformed into triangle $A'B'C'$ by the translation vector (4, 3). If A (2, 1), B (3, 5) and $C(-1, -2)$, find the coordinates of A', B' and C'.
3. Quadrilateral $ABCD$ is transformed into $A'B'C'D'$ by a translation vector (3, –2).
4. If $A(1, 2)$, B (3, 4), $C(7, 4)$ and $D(2, 5)$, then find A', B', C' and D'.
5. If a translation T takes (2, –5) to (–2, 1), find the image of the line ℓ: $2x - 3y = 7$.

If two objects are similar, then they have exactly the same shape, but one of them is an enlargement or reduction of the other without affecting its shape. In scale drawing, the shape of the object and its image are exactly the same, but it is generally larger or smaller than the real object.

1. The concept of similar figures consider only in shape and size relation of the objects.
2. Similar geometric figures are figures which have exactly the same shape but not necessarily the same size.

In the figure below, square $A'B'C'D'$ is the image of square $ABCD$ of side 10 cm under enlargement of scale factor 5. Then $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA} = 5$

$A'B' = B'C' = C'D' = D'A' = 5 \times 10$ cm = 50 cm, which implies that $A'B'C'D'$ is a square of side 50 cm.

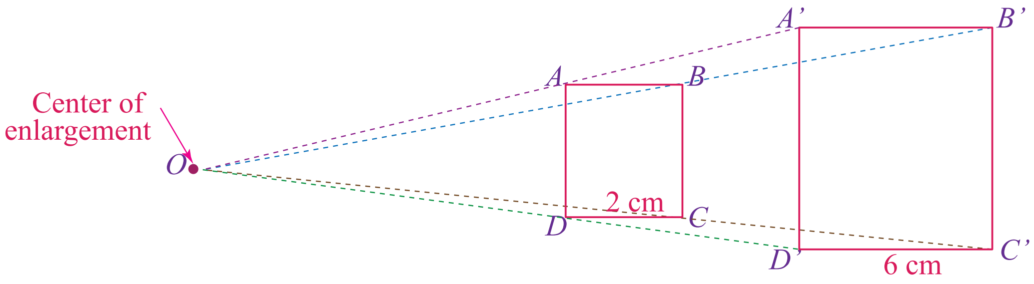


Figure 3.

Scale factor is the number or the conversion factor which is used to change the size of a figure without changing its shape. It is used to increase or decrease the size of an object. Scale factor can be calculated if the dimensions of the original figure and the dimension of the changed (increased or decreased) figures are known. It is given as:

$$\text{Scale factor} = \frac{\text{length of enlarged size}}{\text{length of original}}.$$

Exercises

1. Suppose B and C are plane figures obtained by enlarging and reducing another plane figure A by scale factor 5 and $\frac{1}{5}$ respectively, then
   (a) Are A' A'' and A the same in shape? Why?
   (b) Are and A equal in size? Why?
   (c) What can you say about ratio of sides of these figures?
2. What is the length of the image of a 20cm long segment after enlargement with a scale factor 5?
3. Which members of the following families of plane figure are similar in shape?
   (a) Rectangles
   (b) Parallelograms
   (c) Squares
   (d) Equilateral triangles
   (e) Isosceles triangles
   (f) Rhombus

As the name indicates, reflection transforms an object using a reflecting material.

Definition

Let L be a fixed line in the plane. A reflection M about a line L is a transformation of the plane onto itself which carries each point P of the plane into the point P' of the plane such that L is the perpendicular bisector of the line segment PP'.

The line L is said to be the line of reflection or the axis of reflection

The reflection of point P about the line L, is denoted by $M(P)$, i.e. $P' = M(P)$.

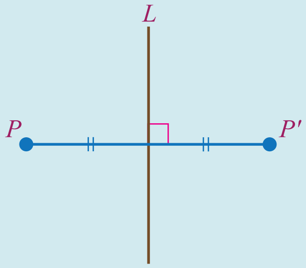


Figure 4.

Note

Every point on the axis of reflection is its own image.

Reflection has the following properties:

1. A reflection about a line L has the property that, if for two points P and Q in the plane, $P = Q$, then $M(P) = M(Q)$. Hence, reflection is a function from the set of points in the plane into the set of points in the plane.
2. A reflection about a line L maps distinct points to distinct points, i.e., if $P \neq Q$, then $M(P) \neq M(Q)$. Equivalently, it has the property that if, for two points P, Q in the plane, $M(P) = M(Q)$, then $P = Q$. Thus, reflection is a one-to-one mapping.
3. For every point P' in the plane, there exists a point P such that $M(P) = P'$. If the point P' is on L, then there exists $P = P'$ such that $M(P) = P'$. Thus, reflection is an onto mapping.

Theorem

A reflection M is a rigid motion. That is, if $P' = M(P)$ and $Q' = M(Q)$, then $PQ = P'Q'$.

We now consider reflections with respect to the axes and the lines $y = mx + b$.

A. Reflection in the line $y = mx$, where $m = \tan\theta$

Let ℓ be a line passing through the origin and making an angle θ with the positive x-axis. Then the slope of ℓ is given by $m = \tan\theta$ and its equation is $y = mx$. See Figure 5.

You will now find the image of a point $P(x, y)$ when it is reflected about this line. *See Figure 6.*

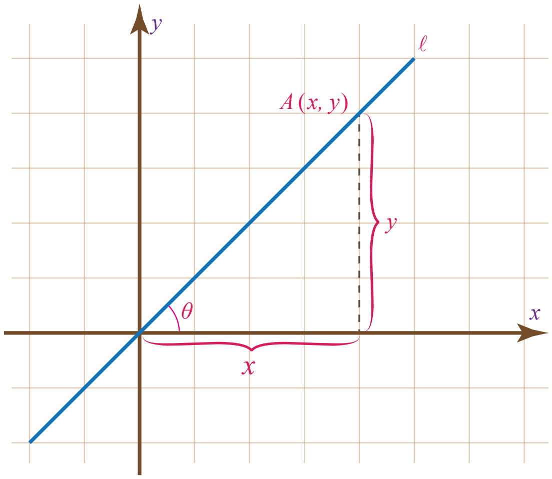


Figure 5.

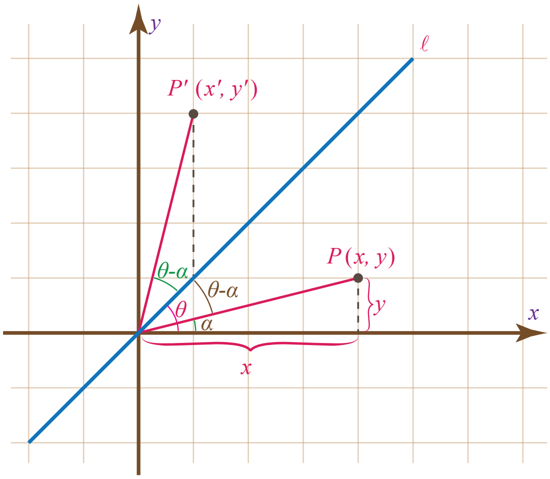


Figure 6.

Let P' (x', y') be the image of P (x, y) after it is reflected along the line $y = mx$.

The coordinates of P are:

$x = r \cos \alpha$ and $y = r \sin \alpha$.

The coordinates of P' are:

$x' = r \cos (2\theta - \alpha)$ and $y' = r \sin (2\theta - \alpha)$.

Expanding $\cos (2\theta - \alpha)$ and $\sin (2\theta - \alpha)$,

Now, use the following trigonometric identities.

1. Sine of the sum and the difference
   - $\sin (x + y) = \sin x \cos y + \cos x \sin y$
   - $\sin (x - y) = \sin x \cos y - \cos x \sin y$.
2. Cosine of the sum and difference
   - $\cos (x + y) = \cos x \cos y - \sin x \sin y$
   - $\cos (x - y) = \cos x \cos y + \sin x \sin y$.

Using these trigonometric identities, you obtain:

$$x' = r[\cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha] = x \cos 2\theta + y \sin 2\theta, = x \cos 2\theta + y \sin 2\theta, \text{ and}$$

$$y' = r[\sin 2\theta \cos \alpha - \sin \alpha \cos 2\theta] = (r \cos \alpha) \sin 2\theta - (r \sin \alpha) \cos 2\theta,$$

$$= x \sin 2\theta - y \cos 2\theta.$$

Thus, the coordinates of P' (x', y'), the image of the point P (x, y) when reflected about the line $y = mx$ is:

$$x' = x \cos 2\theta + y \sin 2\theta \qquad y' = x \sin 2\theta - y \cos 2\theta$$

where θ is the angle of inclination of the line $\ell : y = mx$.

Based on the value of θ, you will have the following four special cases:

1. When θ = 0, you will have reflection in the x-axis. Thus, (x, y) is mapped to $(x, -y)$.
2. When $\theta = \frac{\pi}{4}$, you will have reflection about the line $y = x$ and hence (x, y) is mapped to (y, x).
3. When $\theta = \frac{\pi}{2}$, you will have reflection in the y – axis and (x, y) is mapped to $(-x, y)$.
4. When $\theta = \frac{3\pi}{4}$, you will have reflection about the line $y = -x$ and (x, y) is mapped to $(-y, -x)$.

Example 6

Find the images of the points (3, 2), (0, 1) and (–5, 7) when reflected about the line $y = mx$, where $m = \tan\theta$ and $\theta = \frac{\pi}{4}$.

Solution

This is actually a reflection about the line $y = x$. Thus, the images of (3, 2), (0, 1) and (–5, 7) are (2, 3), (1, 0) and (7, –5), respectively.

Example 7

Find the images of the points P (3, 2), Q (0, 1) and R (–5, 7) when reflected about the line $y = \frac{1}{\sqrt{3}}x$.

Solution

Since $\tan\theta = \frac{1}{\sqrt{3}}$, you have $\theta = \frac{\pi}{6}$. Thus, if $P'(x', y')$ is the image of P, then

$$x' = x\cos 2\theta + y\sin 2\theta = 3\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) = 3\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{2}.$$

$$y' = x\sin 2\theta - y\cos 2\theta = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) - 2\times\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - 1$$

Hence, the image of P (3, 2) is $P'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} - 1\right)$

Similarly, you can show that the images of Q (0, 1) and R (–5, 7) are $Q'\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $R'\left(\frac{-5+7\sqrt{3}}{2}, \frac{-5\sqrt{3}-7}{2}\right)$, respectively.

Example 8

Find the image of $A = (1, -2)$ after it has been reflected in the line $y = 2x$.

Solution

$y = 2x \Rightarrow y = (\tan\theta)\,x \Rightarrow \theta = \tan^{-1}(2)$.

But, from trigonometry, you have $\sin\theta = \dfrac{2}{\sqrt{5}}$ and $\cos\theta = \dfrac{1}{\sqrt{5}}$.

This implies $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \dfrac{1}{5} - \dfrac{4}{5} = -\dfrac{3}{5}$ and $\sin(2\theta) = 2\sin\theta\cos\theta = \dfrac{4}{5}$.

Thus, $x' = -\dfrac{3}{5}x + \dfrac{4}{5}y$ and $y' = \dfrac{4}{5}x + \dfrac{3}{5}y$.

Therefore, $M((1, -2)) = \left(-\dfrac{11}{5}, -\dfrac{2}{5}\right)$.

Note

1. If a line ℓ' is perpendicular to the axis of reflection L, then L' is its own image.
2. If the centre of a circle C is on the line of reflection L, then the image of C is itself.
3. If the centre O of a circle C has image O' when reflected about a line L, then the image circle has centre O' and radius the same as C.
4. If ℓ' is a line parallel to the line of reflection L, to find the image of L' when reflected about L, we follow the following steps.

Step a: Choose any point P on ℓ'.

Step b: Find the image of P, $M(P) = P'$.

Step c: Find the equation of ℓ', which is the line passing through P' with slope equal to the slope of ℓ.

B. Reflection in the line $y = mx + b$

Let $\ell : y = mx + b$ be the line of reflection, where $m \in \mathbb{R}\setminus\{0\}$.

Let $P(x, y)$ be a point in the plane, not on ℓ.

Let $P'(x', y')$ be the image of $P(x, y)$ when reflected about the line ℓ.

Let ℓ' be the line passing through the points $P(x, y)$ and $P'(x', y')$. Then, ℓ' is perpendicular to ℓ, since ℓ is perpendicular to $\overline{PP'}$. Since the slope of ℓ is m, the slope of ℓ' is $-\frac{1}{m}$. Thus, one can determine the equation of the line ℓ'. If A is the point of intersection of ℓ and ℓ', taking A as the midpoint of $\overline{PP'}$, we can find the coordinates of P'.

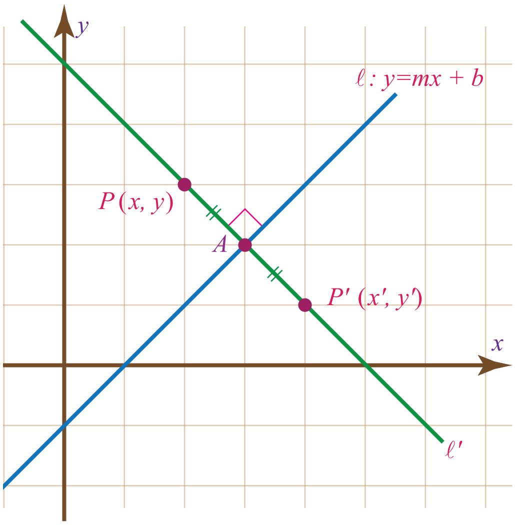


Figure 7.

Thus, to find the image of a point $P(x, y)$ when reflected about a line ℓ, we follow the following four steps.

Step 1: Find the slope of the line ℓ, say m.

Step 2: Find the equation of the line ℓ', which passes through the point $P(x, y)$ and has slope $-\frac{1}{m}$

Step 3: Find the point of intersection A of ℓ and ℓ' which serves as the midpoint of $\overline{PP'}$.

Step 4: Using A as the mid-point of $\overline{PP'}$, find the coordinates of P'.

Example 9

Find images of the following lines after reflection in the line $y = 2x - 3$.

(a) $2y + x = 1$ (b) $y = 2x + 1$ (c) $y = 3x + 4$

Solution

(a) The image of ℓ: $2y + x = 1$ is itself. Explain!

(b) ℓ: $y = 2x + 1$ is parallel to the reflecting axis.

Hence ℓ': $y = 2x + b$. We need to determine b.

Let (a, b) be any point on ℓ, say $(0, 1)$, so that its image lies on ℓ'.

By the above reflecting procedure,

$M((0, 1)) = (a', b')$ $\quad \frac{b'-1}{a'-0} = -\frac{1}{2}$ which implies $a' = -2b' + 2$.

Also, the midpoint of $(0, 1)$ and (a', b') which is $\left(\frac{a'}{2}, \frac{b'+1}{2}\right)$ lies on the reflecting axis

This implies $\frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}$, But $a' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2} \Rightarrow b' = -\frac{3}{5}$ and $a' = \frac{16}{5}$.

This implies the point $\left(\frac{16}{5}, -\frac{3}{5}\right)$ lies on ℓ'.

$$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b \quad \Rightarrow \quad b = -7 \qquad \Rightarrow \ell' : y = 2x - 7.$$

(c) $\ell: y = 3x + 4$ and the axis of reflection $y = 2x - 3$ meet at $(-7, -17)$

Next, take a point on ℓ say $(0, 4)$ and find its image (a', b') so that ℓ passes through (a', b'). Perform the technique similar to the problem in *b*.

Thus, $\frac{b'-4}{a'-0} = -\frac{1}{2}$ and $\frac{4+b'}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{28}{5}$ and $b' = \frac{6}{5}$.

$$\Rightarrow \ \ell' : y = \frac{91}{63}x - \frac{434}{63}.$$

Example 10

Find the image of $(-1, 5)$ when reflected about the lines

(a) $y = -1$

(b) $x = 1$

(c) $y = x + 2$

(d) $y = 2x + 5$

Solution

(a) The image of the point $(-1, 5)$ when reflected about the line $y = -1$ is $(-1, -7)$.

(b) The image of the point $(-1, 5)$ when reflected about the line $x = 1$ is $(3, 5)$.

(c) The slope of $y = x + 2$ is 1.

Let $P'(x', y')$ be the image of $P(-1, 5)$. If ℓ' is the line passing through P and P', then its slope is $\frac{-1}{1} = -1$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1} = -1 \Rightarrow \ell' : y = -x + 4.$$

The point of intersection of ℓ and ℓ' is $(1, 3)$. Taking $(1, 3)$ as a midpoint of $\overline{PP'}$, we get,

$$\frac{-1+x'}{2} = 1 \text{ and } \frac{5+y'}{2} = 3 \ \Rightarrow -1 + x' = 2 \text{ and } 5 + y' = 6$$

$$\Rightarrow x' = 3 \text{ and } y' = 1.$$

Therefore, the image of $P(-1, 5)$ is $P'(3, 1)$.

(d) The slope of $y = 2x + 5$ is 2. If $P'(x', y')$ is the image of $P(-1, 5)$ and ℓ' is the line through P and P', then its slope is $\frac{-1}{2}$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1} = \frac{-1}{2} \Rightarrow \ell' : y = \frac{-1}{2}x + \frac{9}{2}.$$

The point of intersection of ℓ and ℓ' is $A\left(\frac{-1}{5}, \frac{23}{5}\right)$. Taking A as the midpoint of $\overline{PP'}$, find the coordinates of P' as:

$$\frac{-1+x'}{2} = \frac{-1}{5} \text{ and } \frac{5+y'}{2} = \frac{23}{5} \Rightarrow -5 + 5x' = -2 \text{ and } 25 + 5y' = 46.$$

$$\Rightarrow 5x' = 3 \text{ and } 5y' = 46 - 25 = 21 \Rightarrow x' = \frac{3}{5} \text{ and } y' = \frac{21}{5}.$$

Hence, the image of $P(-1, 5)$ is $P'\left(\frac{3}{5}, \frac{21}{5}\right)$.

Exercises

1. The vertices of triangle ABC are $A(2, 1)$, $B(3, -2)$ and $C(5, -3)$. Give the coordinates of the vertices after:
   (a) a reflection in the x-axis
   (b) a reflection in the line $x + y = 0$
   (c) a reflection in the y-axis
   (d) a reflection in the line $y = x$.
2. Find the image of the point $(-4, 3)$ after a reflection about the line $\ell : y = x - 2$
3. If the image of the point $(-1, 2)$ under reflection is $(1, 0)$, find the line of reflection.
4. Find out some of the figures which are their own images in reflection about the line $y = x$.
5. Find the image of the line $\ell : y = x + 4$ after it has been reflected about the line $L : y = x - 3$
6. Find the image of the line $\ell : y = 2x + 1$ after it has been reflected about the line $L : y = 3x + 2$

Key Terms

- Non-rigid motion
- Reflection
- Rigid motion
- Transformation
- Translation

Summary

- **Transformation of the plane**
  (i) Transformation can be classified as rigid motion and non-rigid motion.
  (ii) Rigid motion is a motion that preserves distance. Otherwise it is non-rigid.
  (iii) Identity transformation is a transformation that image of every point is itself.
- **Translation**
  Translation is a transformation in which every point of a figure is moved along the same direction through the same distance.
  (i) Translation vector: If point P is translated to P', the vector $\overrightarrow{PP'}$ is said to be the translation vector.
  (ii) If $\boldsymbol{u} = (h, k)$ is a translation vector, then $T(x, y) = (x + h, y + k)$.
- **Reflection**
  A reflection M about a fixed line L is a transformation of the plane onto itself which maps each point P of the plane into the point P' of the plane such that L is the perpendicular bisector of PP'.
  (i) Point of reflection in the x-axis, $M(x, y) = (x, -y)$
  (ii) Point of reflection in the y-axis, $M(x, y) = (-x, y)$
  (iii) Point of reflection in the line $y = x$, $M(x, y) = (y, x)$
  (iv) Point of reflection in the line $y = -x$, $M(x, y) = (-y, -x)$
  (v) Reflection in the line $y = mx$, $M(x, y) = (x', y')$
  $x' = x \cos 2\theta + y \sin 2\theta$, $y' = x \sin 2\theta - y \cos 2\theta$; $m = \tan \theta$

Exercises

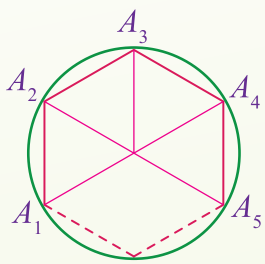
1. If a translation T carries the point (7, –12) to (9, –10), find the images of the following lines and circles.
   (a) $y = 2x - 5$ (c) $x + y = 10$ (e) $x^2 + y^2 - 2x + 5y = 0$
   (b) $2y - 5x = 4$ (d) $x^2 + y^2 = 3$
2. In a reflection the image of the point P (3, 10) is P' (7, 2). Find the equation of the line of reflection.
3. If T is a translation that sends (0, 0) to (3, –2) and M is a reflection that maps (0, 0) to (2, 4), find
   (a) $T(M(1, 3))$ (b) $M(T(1, 3))$
4. In a reflection, the image of the line $y - 2x = 3$ is the line $2y - x = 9$. Find the axis of reflection.

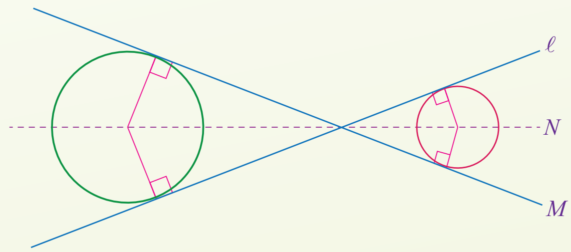
M12CH18

CHAPTER 18

PLANE GEOMETRY

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- define, and discuss polygon;
- define and discuss regular polygons and their properties;
- name and define polygons according to their sides;
- determine interior and exterior angles of polygons and regular polygons;
- find the sum of the interior angles of a polygon and a regular polygon;
- find the sum of the exterior angles of a polygon;
- solve problems on polygons;
- define , discuss isosceles, scalene and equilateral triangles;
- solve problems on isosceles, scalene and equilateral triangles;
- define four quadrilaterals and discuss their properties;
- solve problems on quadrilaterals.

Introduction

A plane is a flat two dimensional surface that extends indefinitely. A plane has zero thickness, infinite length and infinite width.

We often draw a shaded parallelogram to represent a plane with edges, but a plane has no edges.

Some real life examples of a plane are surfaces of a piece of paper, a wall, and a floor.

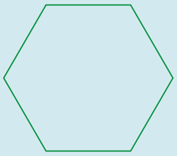
In this unit we learn about figures on a plane or plane figures. A plane figure is a geometric figure that has no thickness. A plane figure lies on only one plane. Plane figure can be made of lines, curves or both.

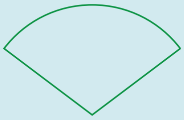
A plane figure can be closed or open.

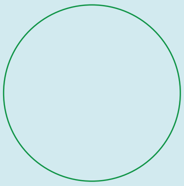
A polygon is a combination of two Greek words poly means many and gon means sides.

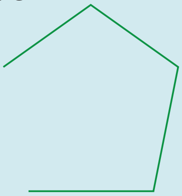
Activity 1

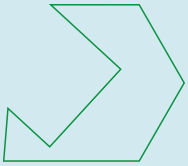
Which of the following figures are polygons?

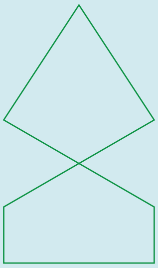

(a)

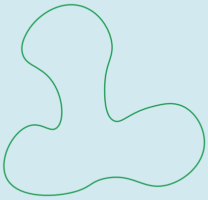

(b)


(c)


(d)


(e)


(f)


(g)

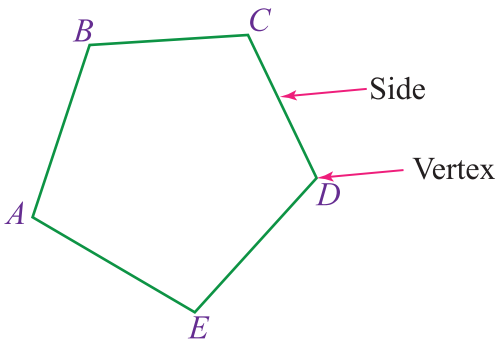
Definition

A polygon is a plane figure made of lines segments connected to each other end to end. The line segments of the polygon are said to be sides or edges. The point where the sides meet is said to be vertex. An angle is formed by adjacent sides of a polygon.

In the figure, $ABCDE$ is a polygon, the sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$.

The vertices are at A, B, C, D and E.

Angles of the polygon are $\angle ABC$, $\angle BCD$, $\angle CDE$, $\angle DEA$ and $\angle EAB$.



Polygons can be classified based on the number of sides, the measure of their sides, the measure of their angles and their boundaries.

1. Types of polygons based on the number of their sides.

Activity 2

1. Copy the following table and draw the polygons under the column of shape.

Name of Polygon	Number of Sides	Shape
Triangle	3	
Quadrilateral	4	
Pentagon	5	
Hexagon	6	
Heptagon	7	
Octagon	8	

Name of Polygon	Number of Sides	Shape
Nonagon	9	
Decagon	10	
Hendecagon	11	
Dodecagon	12	
Triskaidecagon	13	

2. What is the name of polygon has 20 sides?
3. What is the number of sides of hectagon?

Polygons which have more than 13 sides are simply said to be n - gons.

Types of polygons based on their Angles

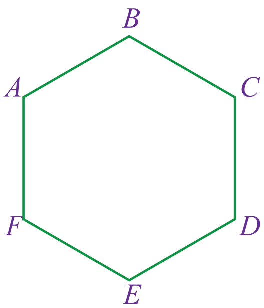
Based on the measure of the interior angles, polygons are classified as convex polygon and concave polygon.

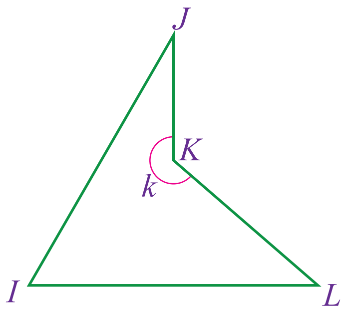
Convex Polygon

A convex polygon is a polygon each of whose interior angles measures less than 180^o.

Concave Polygon

A concave polygon is a polygon which has at least one interior angle that measures more than 180^o.





ABCDEF is a convex polygon.

IJKL is a concave polygon, since interior angle k measures more than 180^o.

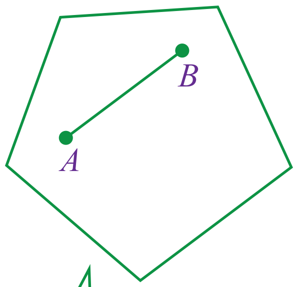
Note: In polygon *IJKL*, $\angle k$ points in the inward direction.

All angles of a convex polygon point in the out wards direction.

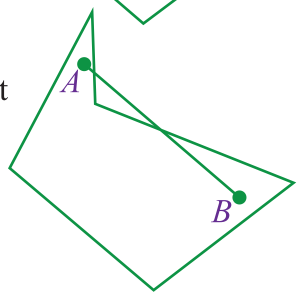
The diagonals of a convex polygon lie inside the polygon.

Diagonal $\overline{JL}$ lies out sides polygon *IJKL*.

If *A* and *B* are any two points inside a convex polygon, then $\overline{AB}$ lies in the interior of the polygon.



There are points *A* and *B* inside a concave polygon such that part of $\overline{AB}$ lies outside the polygon.



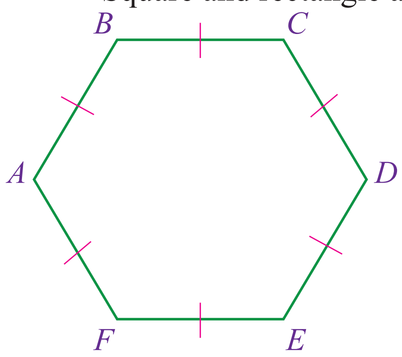
18.3 Types of Polygons based on Measurements of Sides and Angles

Equilateral polygons: An equilateral polygon is a polygon all of whose sides are congruent.

Square and Rhombus are equilateral polygons.

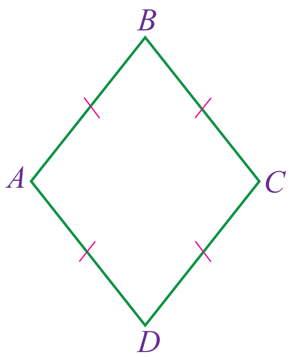
Equiangular polygon: An equiangular polygon is a polygon all of whose angles are congruent.

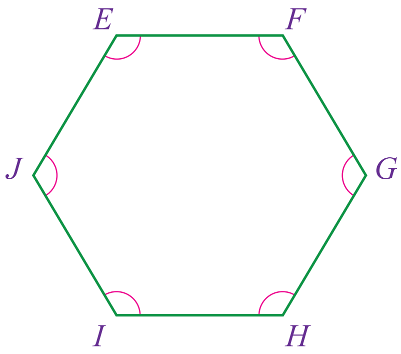
- Square and rectangle are equiangular polygons.



If $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EF} \cong \overline{FA}$, then *ABCDEF* is equilateral.

Rhombus is equilateral.



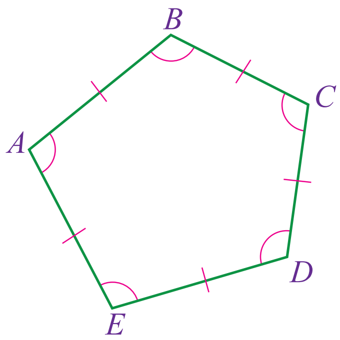


If $\angle E \cong \angle F \cong \angle G \cong \angle H \cong \angle I \cong \angle J$, then *EFGHIJ* is equiangular.

Rectangle is equiangular.

Rectangle

A polygon need not necessarily be equilateral or equiangular. Hence, polygons can be differentiated on the basis of the measurement of their sides and angles. They are classified as a regular and irregular polygons.



Regular polygons: A regular polygon is a polygon that is both equilateral and equiangular.

Equilateral triangle and square are regular polygons.

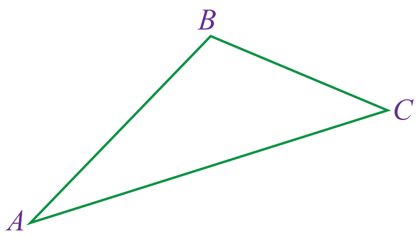
In polygon *ABCDE*, if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EA}$ and $\angle A \cong \angle B \cong \angle C \cong \angle D \cong \angle E$, then *ABCDE* is a regular polygon.

- A regular polygon is convex polygon.

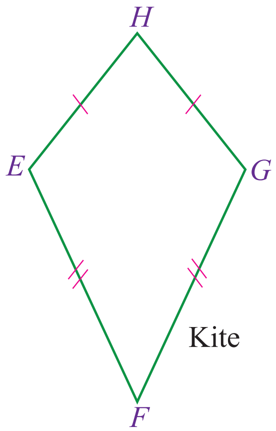
1. Irregular or Non - regular polygons

An irregular polygon is a polygon either the sides are not congruent or the angles are not congruent.

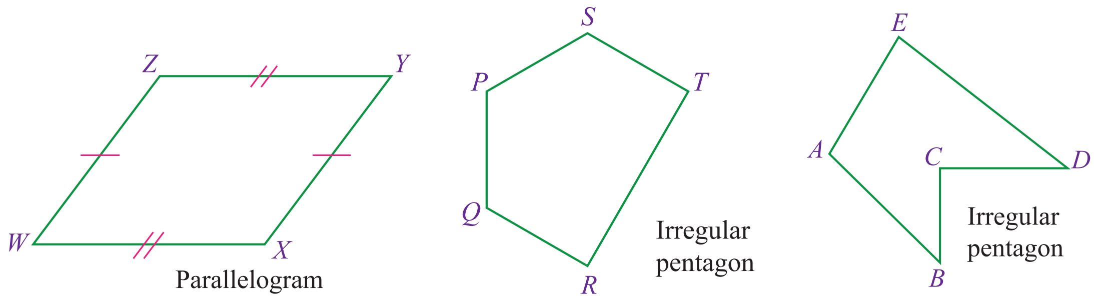
- Scalene triangle and kite are irregular polygons.



Rectangle



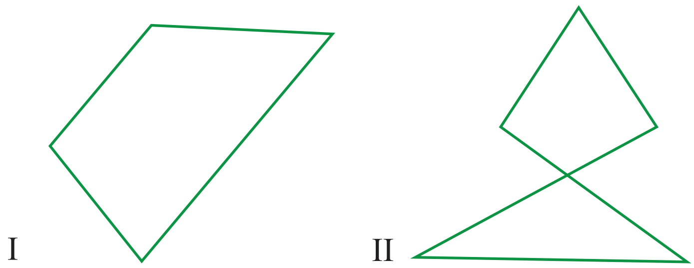
Kite



- Irregular polygons can be concave or convex.

Types of polygons based on their boundaries

Consider the following polygons



Polygon I does not cross its boundary where as polygon II crosses its boundary.

Hence, polygons can be classified based on their boundaries as simple and complex polygons.

Simple polygon: A polygon is simple polygon if it does not intersect itself.

Polygon *I* is simple polygon

- A simple polygon consists of one boundary.
- **Complex polygon:** A polygon is complex polygon if it intersect itself.
- A complex polygon has more than one boundary. Polygon II is complex polygon.
- A polygon can be classified in different types. For example, a trapezium is a quadrilateral, irregular, convex and simple polygon.

The sum of the measures of the interior angles of a polygon.

We know that the interior angle sum of a triangle is 180^{o}. From this, we can see the interior angle sum of any convex polygon.

We can draw only one diagonal from each vertex of a quadrilateral as shown in the figure.

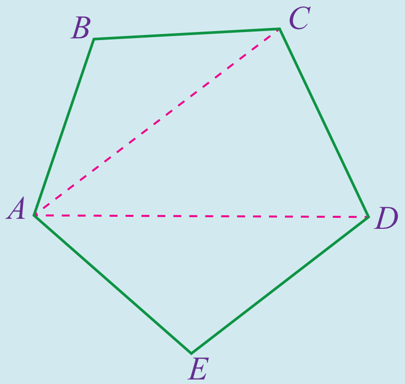
Diagonal $\overline{AC}$ divides *ABCD* in to two triangles and we can see that the sum of the interior angles of a quadrilateral is 2×180^{o} or 360^{o}.

Remark

In this book, by a polygon we mean a convex polygon.

Activity 3

1. Draw diagonals from one vertex of a pentagon and determine the sum of the measures of the interior angles.



2. Copy and fill in the following table and draw a conclusion about the sum of the measures of the interior angles of the polygon.

Number of sides n	Diagonals from one vertex	Number of triangles	Interior angle sum
4	1	2	2×180^{o}
5	2	3	3×180^{o}
6			
7			
8			
20			

Theorem: The sum of the measures of the interior angles of n - sided polygon is $(n - 2) \times 180^{o}$.

Example 1

Find the sum of the degree measures of the interior angles of

(a) Hexagon (b) Octagon

Solution

(a) Hexagon has 6 sides.

Hence, the sum of the degree measures of the interior angles of a hexagon is $(6-2)\times 180^{\circ} = 720^{\circ}$.

(b) Octagon has 8 sides.

Hence, the sum of the degree measures of the interior angles of an octagon is $(8-2)\times 180^{\circ} = 1080^{\circ}$.

Example 2

The interior angles of a pentagon are.

x°, $2x^{\circ}$, $(x+30)^{\circ}$, $(x-10)^{\circ}$ and $(x+40)^{\circ}$.

Find the measure of each interior angle of the pentagon.

Solution

The sum of the measures of the interior angles of a pentagon is 540°.

Thus, $x + 2x + (x+30) + (x-10) + (x+40) = 540$

$\Rightarrow 6x + 60 = 540$

$\Rightarrow 6x = 480$

$\Rightarrow x = 80$

$\Rightarrow 2x = 160$, $x + 30 = 110$, $x - 10 = 70$ and $x + 40 = 120$.

Therefore, the measures of its angles are:

80°, 160°, 110°, 70° and 120°.

Example 3

If the sum of the measures of the interior angles of a polygon is 1980°, then find the number of sides of the polygon.

Solution

Let the number of sides of the polygon be n. Then,

$(n-2)\times 180^{\circ} = 1980^{\circ}$

$$\Rightarrow n - 2 = \frac{1980}{180}$$

$\Rightarrow n - 2 = 11$

$\Rightarrow n = 13$

$\Rightarrow$ The polygon has 13 sides.

Note: The measure of each interior angle of an equiangular polygon of n sides is $\frac{(n-2) \times 180^o}{n}$.

Thus, the measure of each interior angle of a regular polygon of n sides is $\frac{(n-2) \times 180^o}{n}$.

Example 4

Find the measure of each interior angle of a regular.

(a) hexagon (b) octagon

Solution

(a) hexagon has 6 sides

Thus, $\frac{(n-2) \times 180°}{n} = \frac{(6 - 2) \times 180°}{6}$

$= 120^o$.

(b) octagon has 8 sides

Thus, $\frac{(n-2) \times 180°}{n} = \frac{(8 - 2) \times 180°}{8} = 135°$.

Example 5

If each interior angle of a regular polygon measures 175°, then find the number of sides of the polygon.

Solution

$\frac{(n-2) \times 180°}{n} = 175$

$\Rightarrow 180n - 360 = 175n$

$\Rightarrow 180n - 175n = 360$

$\Rightarrow 5n = 360$

$\Rightarrow n = 72$.

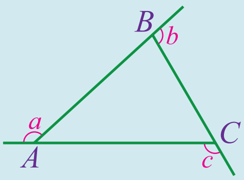
Therefore, the polygon has 72 sides.

The sum of the measures of the Exterior Angles of a Polygon

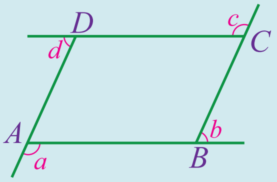
Activity 4

Find the sum of the marked angles in each of the following polygons.

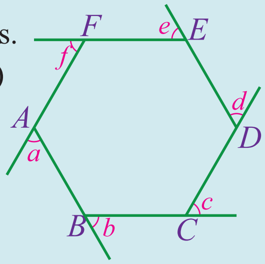
(a)



(b)



(c)



Theorem: The sum of the measures of the exterior angles of a polygon of n sides, taking one angle at each vertex is 360°.

Theorem: The measure of each exterior angle of an equiangular polygon of n sides is $\dfrac{360°}{n}$.

Example 6

Find the measure of each exterior angle of a regular hexagon.

Solution

Hexagon has 6 sides.

Thus, $\dfrac{360°}{n} = \dfrac{360°}{6} = 60°$

Therefore, the measure of each exterior angle of a regular hexagon is 60°.

Example 7

The measure of each interior angle of a regular polygon is 162°, find the number of sides of the polygon.

Solution

At each vertex of a polygon, the interior and exterior angles are supplementary angles as shown in the figure.

180° – 162° = 18°.

Hence, the exterior angle measures 18°.

$$\Rightarrow \frac{360°}{n} = 18°$$

$$\Rightarrow n = \frac{360}{18}$$
$$= 20$$

Therefore, the polygon has 20 sides.

Note: The measure of each interior angles of a regular polygon is

$$180^{\circ} - \frac{360^{\circ}}{n}$$
$$= \frac{n \times 180^{\circ} - 360^{\circ}}{n}$$
$$= \frac{(n-2) \times 180^{\circ}}{n}.$$

Exercises

1. Find the sum of the degree measures of the interior angles of a convex polygon with

   (a) 14 sides (b) 41 sides (c) 33 sides (d) 117 sides

2. The measures of the interior angles of convex polygon are $\frac{2}{5}x^{\circ}$, $(x+20)^{\circ}$, $(3x-10)^{\circ}$, 20°, $4x^{\circ}$ and $\frac{8}{5}x^{\circ}$.
   Find x.

3. Find the measure of an interior angle of a regular polygon whose number of sides are

   (a) 6 (c) 20 (e) 15
   (b) 12 (d) 8

4. Find the measure of each exterior angle of a regular polygon with

   (a) 20 sides (c) 1080 sides (e) 36 sides
   (b) 90 sides (d) 18 sides

5. Find the number of sides of a regular polygon if one of its central angle measures 9°.

6. Find the number of sides of a regular polygon if the measure of each interior angle is

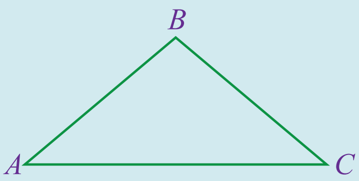
   (a) $\left(\frac{900}{7}\right)^{\circ}$ (b) $\left(\frac{1620}{11}\right)^{\circ}$ (c) $\left(\frac{1080}{7}\right)^{\circ}$

7. Can you have a regular polygon in which the measure of each of its interior angles is

(d) 157.5^o?	(f) 111^o?	(h) 162^o?
(e) 178^o?	(g) 179^o?	

DEFINITION

A triangle is a polygon that has three sides. The polygon shown below is a triangle.



Triangle ABC is denoted by ΔABC.

In ΔABC, each of the points A, B, and C is a vertex of the triangle.

The line segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ are the sides of the triangle.

- Any three non - collinear points determine a unique triangle. There is one and only one triangle whose vertices are three non - collinear points.

Classification of Triangles

ACTIVITY 5

Suppose the lengths of sides of a triangle are 5, n and k units, where n and k are positive integers less than 10.

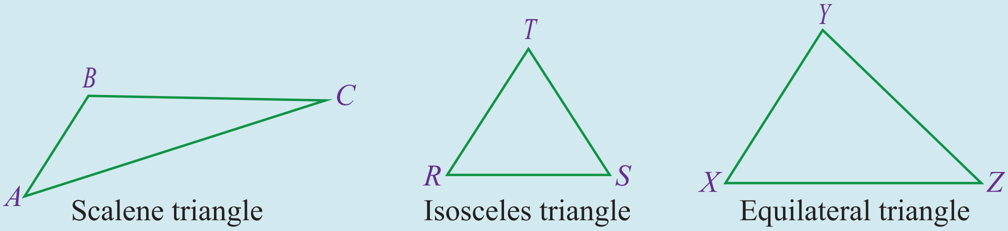
1. Find the values of n and k?
2. Find the number of triangles having
   (a) 3 equal sides (b) 2 equal sides (c) No equal sides

Classifying Triangles Based on Sides

Based on their sides triangle can be classified as scalene triangle, isosceles triangle and equilateral triangle.

DEFINITION

1. A scalene triangle is a triangle that has no congruent sides.
2. An isosceles triangle is a triangle that has two congruent sides.
3. An equilateral triangle is a triangle that has three congruent sides.



Scalene triangle — $AB < BC < AC$

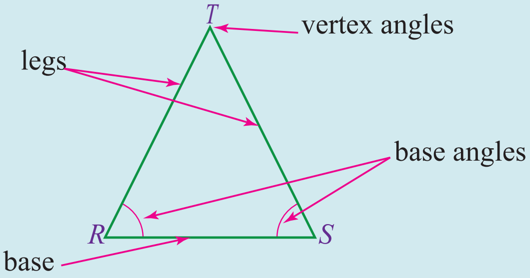
Isosceles triangle — $\overline{TR} \cong \overline{TS}$

Equilateral triangle — $\overline{XY} \cong \overline{YZ} \cong \overline{ZX}$

Definition

- An isosceles triangle, the two congruent sides $\overline{TR}$ and $\overline{TS}$ are called the legs of the triangle.

  The third side $\overline{RS}$, is called the base. The angle opposite to the base, $\angle T$ is called vertex angle.
- The angles opposite to the congruent sides, $\angle R$ and $\angle S$ are called the base angles.



Classifying Triangles Based on Angles

Activity 6

Suppose the measures of the angles of a triangle are 60°, a° and b°, where a and b are multiples of 10.

1. Find the values of a and b.
2. How many triangles have

   (a) 3 congruent angles?

   (b) 2 congruent angles?

   (c) Non congruent angles?

   (d) one right angle?

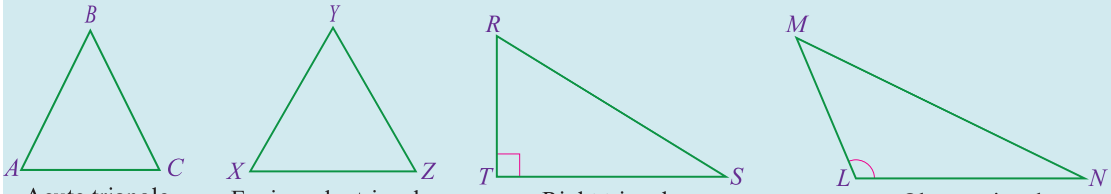
   (e) one obtuse angle?

   (f) 3 acute angles?

Based on their angles triangles can be classified as acute, equiangular, right, and obtuse triangle.

DEFINITION

1. An acute angle triangle is a triangle that has three acute angles.
2. An equiangular triangle is a triangle that has three congruent angles.
3. A right triangle is a triangle that has a right angle.
4. An obtuse triangle is a triangle that has an obtuse angle.



Acute triangle | Equiangular triangle $\angle x \cong \angle y \cong \angle z$ | Right triangle $m(\angle T) = 90^{o}$ | Obtuse triangle $m(\angle L) > 90$

Properties of Equilateral Triangle

- The measure of each angle of an equilateral triangle is 60^{o}.
- The perpendicular drawn from a vertex of an equilateral triangle bisects the vertex angle and the base.
- The medians, angle bisectors, the perpendicular bisectors of sides and the altitudes intersect at the same point.

Concurrent Lines in a Triangle

- Three or more lines are concurrent if they pass through the same point.

The median of a triangle is a line segment joining the vertex of the triangle to the midpoint of its opposite side.

Centroid of a Triangle

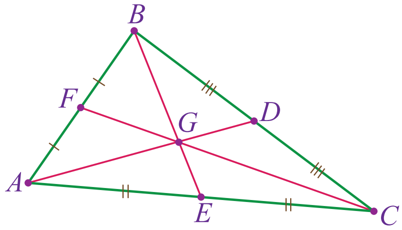
The medians of a triangle intersect at a point called the centroid which is two - thirds of the distance from the vertices to the midpoint of the opposite sides.

ACTIVITY 7

Draw the medians, altitudes, angle bisectors and perpendicular bisectors of the sides of a triangle when the triangle is

(a) scalene
(b) isosceles
(c) equilateral

Let G be the centroid of ΔABC, then,



$$AG = \frac{2}{3}AD \text{ and } DG = \frac{1}{3}AD$$

$$BG = \frac{2}{3}BE \text{ and } EG = \frac{1}{3}BE$$

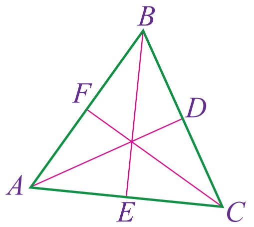
$$CG = \frac{2}{3}CF \text{ and } FG = \frac{1}{3}CF$$

Properties of Medians of a triangle

In an equilateral triangle, the medians are congruent.

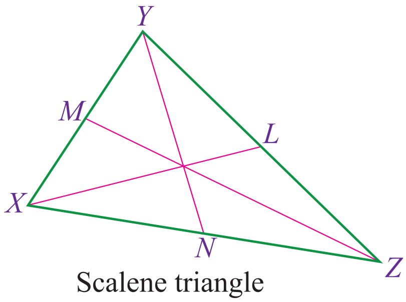
In a scalene triangle, the medians are not congruent to each other.

In an isosceles triangle, the medians from the base angles are congruent.



Equilateral triangle

$\overline{AD} \cong \overline{BE} \cong \overline{CF}$

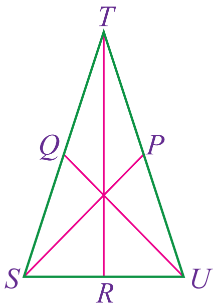


Scalene triangle

$ZM \neq XL$

$ZM \neq YN$

$YN \neq ZM$



Isosceles triangle

$\overline{SP} \cong \overline{UQ}$

EXERCISES

1. Show that a median of a triangle divides the triangle into two triangles of equal area.
2. Show that the three medians of a triangle divide the triangle into 6 triangles of equal areas.

Incenter of a Triangle

The bisectors of the angles of a triangle intersect at a point called in center that is equidistant from sides of the triangle.

The incenter of a triangle is the center of the circle inscribed in the triangle.

The Radius of the circle inscribed in a triangle of perimeter P and area A is $\frac{2A}{P}$.

$$\Rightarrow r = \frac{2A}{p}.$$

EXAMPLE 8

The lengths of the sides of a triangle are 6 cm, 8 cm and 10 cm, find the radius of the triangle inscribed in the triangle.

Solution

ΔABC is a right angel triangle sine $6^2 + 8^2 = 10^2$

The area of $\Delta ABC = \frac{1}{2} \times 6 \times 8 \text{ cm}^2 = 24 \text{ cm}^2$

The perimeter of ΔABC

$= (6 + 8 + 10)$ cm

$= 24$ cm

Therefore, the radius of the circle, $= \frac{2A}{p} = \frac{2 \times 24}{24} \text{cm} = 2$ cm.

Circumcenter of a Triangle

The perpendicular bisectors of the sides of a triangle intersect at appoint called the circum center that is equidistant from the vertices of the triangle.

The circum center of a triangle is the center of the circle circumscribing the triangle.

EXERCISES

If G in the circumcenter of ΔABC, then show that $m(\angle AGB) = 2\ m(\angle C)$ if $\angle C$ is acute and $m(\angle AGB) = 2\ (180^{o} - m(\angle C))$ if $\angle C$ is obtuse.

Orthocenter of a Triangle

The altitudes of a triangle intersect at a point called ortho center.

A quadrilateral is a polygon which has four sides.

Activity 8

1. Draw a quadrilateral which has one pair of opposite sides parallel.
2. Draw a quadrilateral which has both pairs of opposite sides parallel.
3. Draw a quadrilateral which has congruent diagonals.
4. Draw a quadrilateral which has perpendicular diagonals.
5. Draw a quadrilateral which has two pairs of opposite sides congruent.

Trapezium and Parallelogram

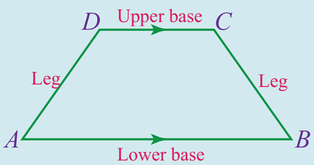
Trapezium

Definition

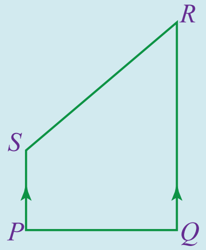
A trapezium is a quadrilateral which has two and only two sides parallel.

Quadrilateral $ABCD$ is a trapezium if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are not parallel.

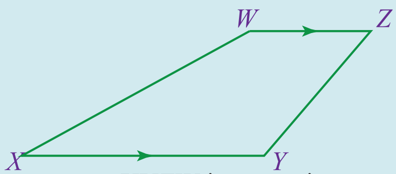
The parallel sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are said to be bases of the trapezium.



ABCD is a tapezium



PQRS is a tapezium



XYZW is a tapezium

The non - parallel sides $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are said to be the legs of the trapezium.

Isosceles Trapezium

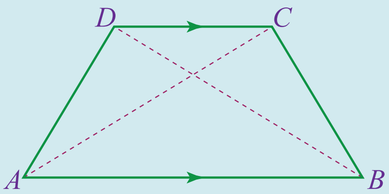
Definition

An isosceles trapezium is a trapezium in which the non - parallel sides are congruent. In trapezium $ABCD$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if $\overleftrightarrow{BC} \cong \overleftrightarrow{AD}$, then $ABCD$ is an isosceles trapezium.

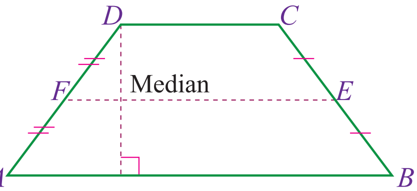
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \cong \overleftrightarrow{BC}$

In an isosceles trapezium, the base angles are congruent and the diagonals are congruent

$\angle A \cong \angle B$ and $\overline{AC} \cong \overline{BD}$



The line segment which joins the midpoints of the non - parallel sides of the trapezium is said to be the median of the trapezium.



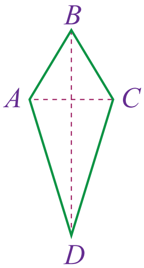
$$\overleftrightarrow{EF} \parallel \overleftrightarrow{AB} \text{ and } \overleftrightarrow{EF} \parallel \overleftrightarrow{CD} \text{ and } EF = \frac{AB + CD}{2}.$$

The median of a trapezium is parallel to the bases. The length of the median of a trapezium is one half the sum of the lengths of the bases.

The area of a trapezium is equal to the product of the length of its altitude and the length of its median.

Kite

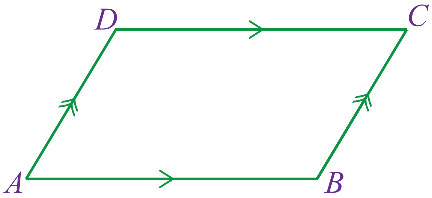
Kite is a quadrilateral which has two pairs of adjacent sides congruent.



$ABCD$ is kite if $\overline{AD} \cong \overline{CD}$ and $\overline{AB} \cong \overline{BC}$.

- The diagonals of a kite are perpendicular.
- The diagonals of a kite bisect the vertex angles.
- The area of a kite is one half the product of the diagonals.

Parallelogram



A parallelogram is a quadrilateral in which both pairs of opposites sides are parallel.

In quadrilateral $ABCD$, if $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, then $ABCD$ is a parallelogram.

Theorems on a Parallelogram

ACTIVITY 9

Draw a diagonal of a parallelogram

What is the relationship between the triangles formed by the diagonal.

What is the relationship between the diagonals.

1. Measures the lengths of opposite sides and opposite angles
2. Compare the triangles formed by the diagonals and the sides of the parallelogram.
3. Find the sum of consecutive angles.

Activity 10

Draw a quadrilateral such that
1. both pairs of opposite sides are congruent.
2. one pair of opposite sides are parallel and congruent.
3. diagonals bisect each other.
4. both pairs of opposite angles are supplementary.

Decide whether or not the quadrilaterals are parallelograms.

Activity 11

Identify the type of parallelogram. When
1. one of its interior angle is 90^{o}
2. adjacent sides are congruent.
3. diagonals are congruent.
4. diagonals are perpendicular.

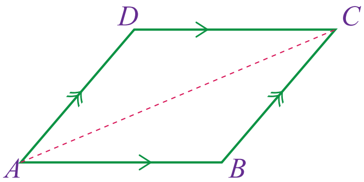
Theorem

1. A diagonal divides a parallelogram into two congruent triangles.

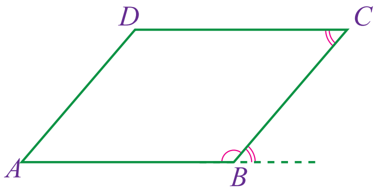
$$\Delta ABC \cong \Delta CDA$$

$$\overline{AB} \cong \overline{CD} \text{ and } \overline{BC} \cong \overline{DA}$$

$$\angle B \cong \angle D \text{ and } \angle A \cong \angle C$$

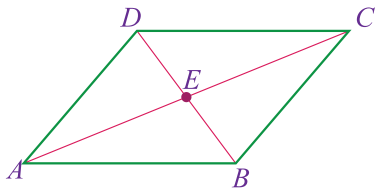


2. Both pairs of opposite sides of a parallelogram are congruent.
3. Both pairs of opposite angles of a parallelogram are congruent.
4. Consecutive angles of a parallelogram are supplementary.



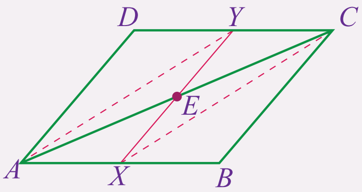
5. The diagonals of a parallelogram bisect each other.

$$\overline{AE} \cong \overline{CE} \text{ and } \overline{BE} \cong \overline{DE}$$



Example 9

Let $ABCD$ be a parallelogram and the line $\overline{XY}$ passes through the midpoint of AC where X and Y are points on $\overline{AB}$ and $\overline{CD}$ respectively. Show that $\overline{AC}$ bisects $\overline{XY}$

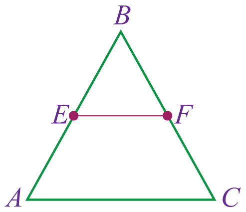


$\Delta AEX \cong \Delta CEY$ by AAS

$\Rightarrow \overline{AX} \cong \overline{CY}$ and $\overline{AX} \parallel \overline{CY}$

$\Rightarrow AXCY$ is a parallelogram

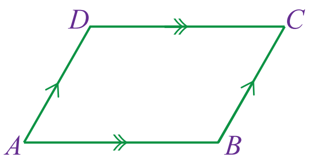
$\Rightarrow$ The diagonals $\overline{AC}$ and $\overline{XY}$ bisect each other.



Theorem: A line segment that joins the mid points of two sides of a triangle is parallel and half as long as the third side.

In ΔABC, if $\overline{AE} \cong \overline{BE}$ and $\overline{BF} \cong \overline{CF}$, then $\overline{EF} \parallel \overline{AC}$ and $EF = \frac{1}{2}AC$.

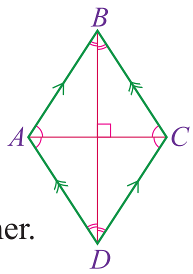
Rhombus: A rhombus is a parallelogram which has two congruent consecutive sides. If $ABCD$ is a parallelogram such that $\overline{AB} \cong \overline{AD}$, then parallelogram $ABCD$ is a rhombus.



$\overline{AB} \cong \overline{BC}$

$\overline{AB} \cong \overline{BC} \cong \overline{DC} \cong \overline{AD}$

Properties of a Rhombus



1. A rhombus has all properties of a parallelogram.
2. A rhombus is equilateral.
3. The diagonals of a rhombus are perpendicular to each other.
4. The diagonals of a rhombus bisects its angles.

Rectangle: A rectangle is a parallelogram one of whose angles is right angle.

$\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$ and $m(\angle A) = 90^o$.

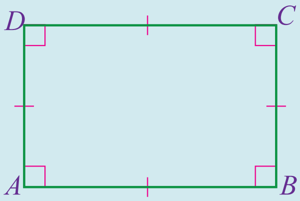
Properties of a Rectangle

- A rectangle is a parallelogram, thus it has all properties of a parallelogram.
- A rectangle is equiangular.
- The diagonals of a rectangle are congruent.

Square

Definition

A square is a rectangle which has two congruent consecutive sides.



In rectangle *ABCD*, if $\overline{AB} \cong \overline{BC}$, then *ABCD* is a square.

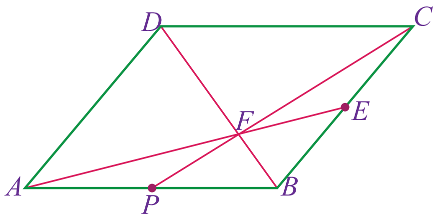
Properties of a square

1. A square has all properties of a rectangle.
2. A square has all properties of a rhombus.
3. A square is equiangular and equilateral quadrilateral.
4. The diagonals of a square are congruent and perpendicular.
5. The diagonals of a square bisect the vertex angles.

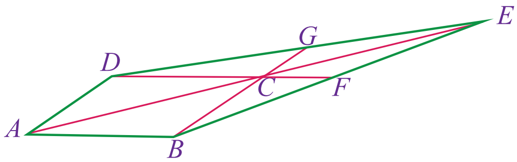
Exercises

1. Suppose $\overline{AJ}$, $\overline{BK}$ and $\overline{CI}$ are medians of ΔABC with centroid at G. If J, K and I are points on the sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively, show that G is the centroid of ΔIJK.

2. In the figure below, $ABCD$ is a parallelogram such that P is midpoint of $\overline{AB}$. Show that E is midpoint of $\overline{BC}$.



3. In the figure $ABCD$ is a parallelogram and C is midpoint of $\overline{AE}$. Show that F is midpoint of $\overline{BE}$ and G is midpoint of $\overline{DE}$.



4. If the altitudes of ΔABC intersect at D, show that the altitudes of ΔDBC intersect at A.
5. Find the radius of the circle that is inscribed in ΔABC when $AB = 6$ cm, $AC = 8$ cm and $BC = 10$ cm.

Regular Polygons and the Circle

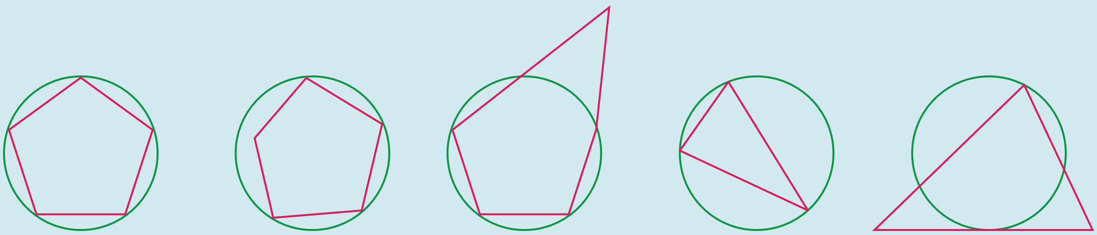
We have seen that a circle can be inscribed and circumscribed about any triangle.

A circle is circumscribed about a polygon if the circle passes through all vertices of the polygon.

Circumcircle and in Circle

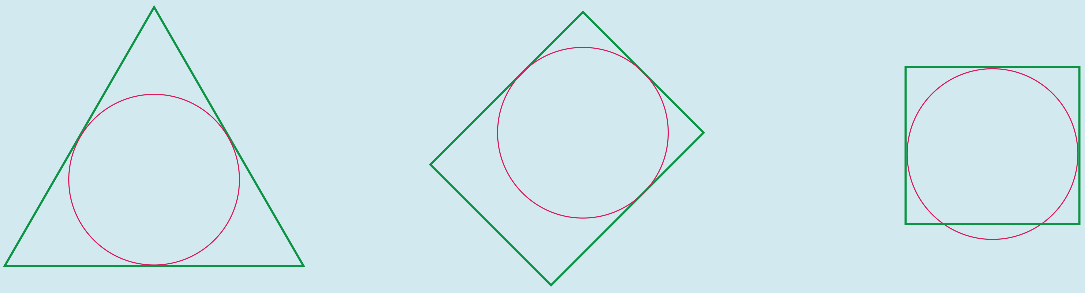
Activity 12

1. Identify the circle that passes through all vertices of the polygon.



(a) (b) (c) (d) (e)

2. Identify the circle that touches every side of the polygon at exactly one point.

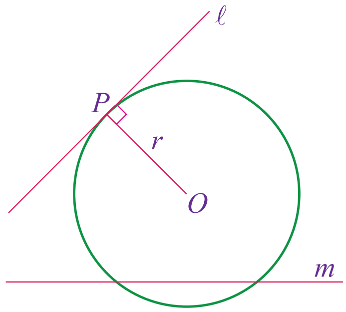


(a) (b) (c)

Tangent lines and Secant lines

A line is said to be **tangent to** a circle if it touches the circle at exactly one point.

A line is said to be **secant to** the circle if it crosses the circle at two different points.



ℓ is tangent to the circle at P.

P is called point of tangency.

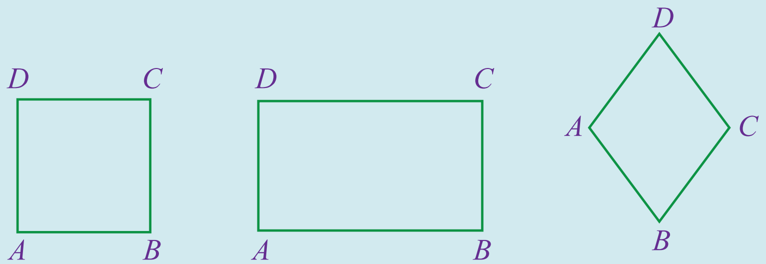
The radius is perpendicular to the tangent line at the point of tangency.

m is secant line.

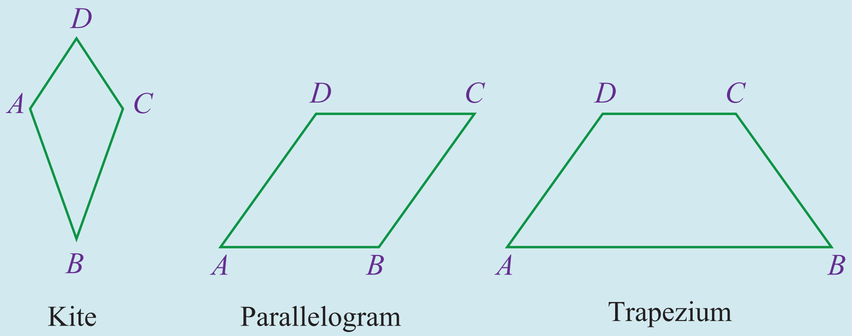
There is exactly one tangent line at a point to a circle.

Activity 13

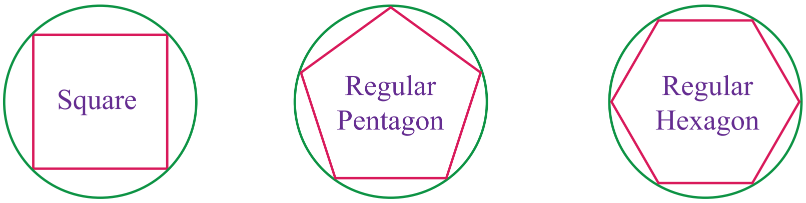
Identify the polygon in which you can inscribe or circumscribe a circle.

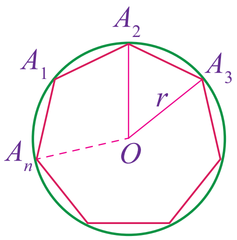


Square Rectangle Rhombus



A circle can be circumscribed about any regular polygon.





Suppose $A_1A_2A_3 \ldots A_n$ is a regular polygon.
which is inscribed in a circle of radius r whose center is at O.

Definition

The radius of the circle is the radius of the circumscribed circle.

The radius of the polygon is OA_i; $i = 1, 2, 3, \ldots n$.

Where $OA_i = OA_j = r$ for all $i, j = 1, 2, 3, \ldots, n$

$\overline{OA_i}$ is the bisector of $\angle A_i$.

Central Angle: A central angle of a regular polygon is an angle formed by two radii of the polygon drawn to consecutive vertices of the polygon.

$\angle A_1OA_2$ is central angle

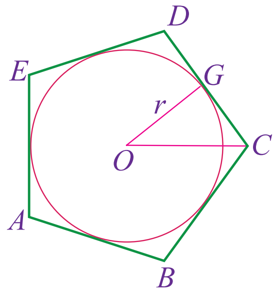
A circle is inscribed in a polygon if every side of the polygon is tangent to the circle.
A circle can be inscribed in a regular polygon.

ABCDE is a regular polygon. A circle of radius *r* whose center at *O* is inscribed in the pentagon.

$\overline{OC}$ is the radius of the pentagon.

An apothem of a regular polygon is the perpendicular bisector of the side of the polygon to which it is drawn. $\overline{OG}$ is an apothem of *ABCDE*.

An apothem of a regular polygon is a radius of its inscribed polygon.

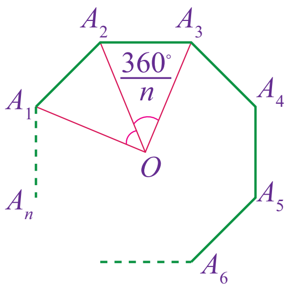


The Measure of a Central Angle of a Regular Polygon

The central angles of a regular polygon are congruent.

Suppose $A_1A_2A_3 \dots A_n$ is a regular polygon that has *n* sides.

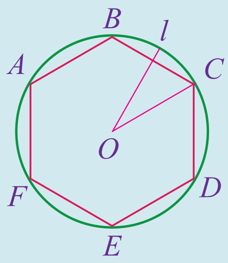
$\angle A_1OA_2$, $\angle A_2OA_3$, …, and $\angle A_nOA_1$ are the central angles.



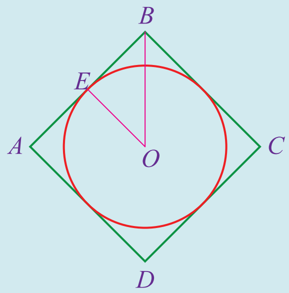
The measure of a central angle of a regular polygon of *n* sides is $\dfrac{360^\circ}{n}$.

Definition

Circumcircle is the circle that passes through all vertices of the polygon.
Incircle is the circle in which all sides of the polygon are tangent to the circle.



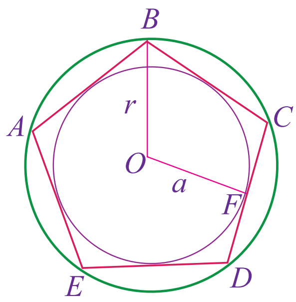
The circle circumscribes the polygon
$\overline{OC}$ is the radius of the circle



The circle is inscribed in the polygon.
$\overline{OE}$ is the radius of the circle

Radius and Apothem of a Regular Polygon

A regular polygon has a radius and center which is the radius and center of the circumscribed circle.



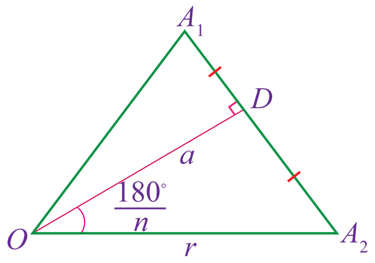
If $ABCDE$ is a regular pentagon, then $\overline{OB}$ is its radius.

The radius of the inscribed circle is $\overline{OF}$.

The radius of a regular polygon is the radius of the circumscribed circle.

The apothem of a regular polygon is the radius of the inscribed circle.

An apothem of a regular polygon is the perpendicular bisector of the corresponding side of the polygon and it is the bisector of the corresponding central angle.



$\overline{OD}$ is the apothem, it bisects $\angle A_1OA_2$.

Hence $m(\angle A_2OD) = \frac{1}{2}m(\angle A_1OA_2)$

$= \frac{1}{2}\left(\frac{360^\circ}{n}\right)$

$= \frac{180^\circ}{n}$

From trigonometric ratio, cos $\angle DOA_2 = \frac{a}{r}$.

Hence, $a = r \angle\cos DOA_1$

$$a = \text{r} \cos \frac{180^\circ}{n}$$

Let $S = A_1 A_2$, then $A_2D = \frac{S}{2}$.

Using trigonometric ratio we have, $\sin\frac{180^\circ}{n} = \frac{A_2D}{r}$

Perimeter of Regular Polygon

The total distance of the outor sides of a closed figure is said to be the perimeter.

The perimeter of a polygon is the sum of the length of all sides of the polygon.

A regular polygon has all sides equal. Therefore, the perimeter of a regular polygon of n sides each side equal to s is ns.

$$\Rightarrow A_2D = r\sin\frac{180^\circ}{n}$$

$$\Rightarrow \frac{s}{2} = r\sin\frac{180^\circ}{n}$$

$$\Rightarrow s = 2r\sin\frac{180^\circ}{n}.$$

The length of each side of a regular polygon that has n side is $2r\sin\frac{180^\circ}{n}$.
The perimeter P of a regular polygon that has n sides is $P = ns$.

Thus, $P = 2nr\sin\frac{180^\circ}{n}$.

Example 10

Find the perimeter of
(a) an equilateral triangle whose radius is 3 cm.
(b) a regular pentagon whose radius is 4 cm.
(c) a regular hexagon whose radius is 8 cm.

Solution

(a) $p = 2nr\sin\frac{180^\circ}{n}$ where $n = 3$ and $r = 3$ cm

Hence, $p = 2 \times 3 \times 3 \sin\frac{180^\circ}{n}$ cm

$= 18 \sin 60^\circ$ cm

$= 18 \times \frac{\sqrt{3}}{2}$ cm

$= 9\sqrt{3}$ cm.

(b) $p = 2nr\sin\frac{180^\circ}{n}$ where $n = 5$ and $r = 4$ cm

Thus, $p = 2 \times 5 \times 4 \sin\frac{180^\circ}{n}$ cm

$= 40 \sin 36^\circ$ cm.

(c) $p = 2nr\sin\frac{180^\circ}{n}$ where $n = 6$ and $r = 8$ cm

Thus, $p = 2 \times 6 \times 8 \sin \frac{180^\circ}{n}$ cm

$= 96 \sin 30^\circ$ cm

$= 96 \times \frac{1}{2}$ cm

$= 48$ cm.

Example 11

Show that the radius of a regular hexagon is equal to the length of its side.

Proof

$s = 2r \sin \frac{180^\circ}{6}$, where $n = 6$.

Then $s = 2r \sin \frac{180^\circ}{6}$

$= 2r \sin 30^\circ$

$= 2r \times \frac{1}{2}$

$= r$

$\Rightarrow s = r$.

The Area of Regular Polygon

Activity 14

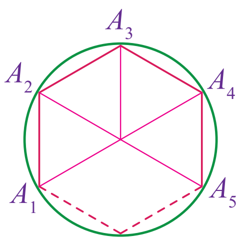
Draw the radii from the center to the vertices of a regular polygon with number of sides $n = 3, 4, 5$ and 6.

(a) How many triangles are formed in each polygon?

(b) What type of triangles are formed?

(c) Using an apothem a as the height of a triangle so formed and the corresponding side of the polygon as the base of the triangle, find the area of the triangle.

(d) Find the sum of the areas of all triangles in each polygon.



Let $A_1A_2A_3 \ldots A_n$ be a regular polygon.

Let the length of each side be s.

Thus, $A_1A_2 = A_2A_3 = \ldots = A_nA_1 = s$

There are n isosceles triangles.

The area of one of the n triangles is $= \frac{1}{2}sa$

Hence the area of a regular polygon of n sides is $\frac{1}{2}Pa$

Example 12

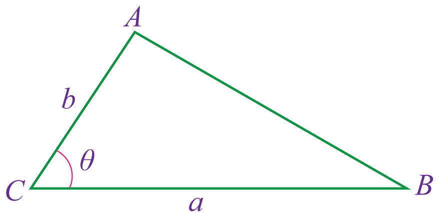
Find the apothem of a square whole radius is 5cm.

Solution

$a = r\cos\dfrac{180^{o}}{n}$ where $n = 4$ and $r = 5$ cm

$a = 5\cos\dfrac{180^{o}}{4}\text{cm} = 5\cos 45^{0}\text{ cm} = \dfrac{5\sqrt{3}}{2}\text{cm}$

Note: The area of a triangle with two sides a and b and the measure of an included angle θ is $\dfrac{1}{2}ab\sin\theta$.

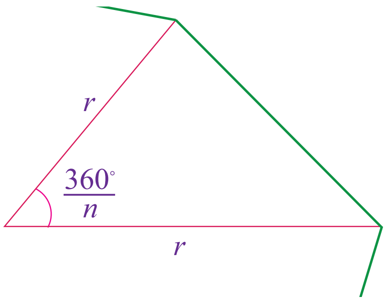


Hence, in a regular polygon of n sides, one of the triangle has an area,

$$\frac{1}{2}r^{2}\sin\frac{360^{o}}{n}.$$

The area of a regular polygon of n sides is

$$A = \frac{1}{2}nr^{2}\sin\frac{360^{o}}{n}.$$



Example 13

Find the area of a regular dodecagon interms of its radius r.

Solution

$A = \dfrac{1}{2}nr^{2}\sin\dfrac{360^{o}}{n}$, where $n = 12$.

Thus, $A = \dfrac{1}{2}\times 12r^{2}\sin\dfrac{360^{o}}{12}$

$= 6r^{2}\sin 30^{o}$

$= 6r^{2}\times\dfrac{1}{2}$

$= 3r^{2}$.

Example 14

Find the area of a regular hexagon whose radius is 4 cm.

Solution

$A = \frac{1}{2}nr^2 \sin\frac{360^0}{n}$, where $n = 6$ and $r = 4$ cm

$\Rightarrow \quad A = \frac{1}{2}\times 6\times(4\text{cm})^2 \sin\frac{360^0}{6}$

$\Rightarrow \quad A = 48 \sin 60^0 \text{ cm}^2$

$= 48 \times \frac{\sqrt{3}}{2}\text{cm}^2 = 24\sqrt{3}\text{cm}^2$.

Example 15

Find the area of an equilateral triangle whose radius is 12 cm.

Solution

$A = \frac{1}{2}nr^2 \sin\frac{360^0}{n}$, where $n = 3$, $r = 12$ cm

$\Rightarrow A = \frac{1}{2}\times 3\times(12\text{cm})^2 \sin\frac{360^0}{3}$

$= \frac{1}{2}\times 3\times 144\times\sin 120^0\text{cm}^2 = 3\times 72\times\frac{\sqrt{3}}{2}\text{cm}^2 = 108\sqrt{3}\text{cm}^2$.

Note: The area of an equilateral triangle is $\frac{3r^2\sqrt{3}}{4}$ or $\frac{S^2\sqrt{3}}{4}$.

Example 16

Find the area of an octagon whose radius is 3 cm.

Solution

$A = \frac{1}{2}nr^2 \sin\frac{360^o}{n}$, where $n = 8$ and $r = 3$ cm

Thus,

$A = \frac{1}{2}\times 12\times 3^2 \sin\frac{360^o}{12}\text{cm}^2 = 54 \sin 30^0 \text{ cm}^2 = 54\times\frac{1}{2}\text{cm}^2 = 27\text{cm}^2$.

Exercises

1. Copy and complete the following table.

Numbers of sides of a regular polygon	radius	side length	perimeter	Area
3	2			
3				$\frac{81\sqrt{3}}{4}$
4				2
6		8		
8				24
12		2		

2. Find the pemeter and area of a regular polygon in terms of the radius r when the number of sides is

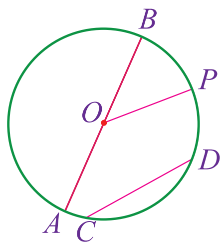
   (a) 3 (b) 4 (c) 8 (d) 10

3. Find the area of a regular hexagon circumscribed about a circle of radius 10 cm.
4. Find the area of a regular hexagon inscribed in a circle of radius 10 cm.

Circle Theorems(I)

A circle is the locus of all points in a plane which are equidistant from a fixed point. The fixed point is called the center of the circle. The distance between the center and any point on the circle is called radius.

Chord of a Circle



A chord of a circle is a line segment joining any two points on the circle.

O is the center, OP is radius, $\overline{CD}$ and $\overline{AB}$ are chords of the circle.

AB is a diameter of the circle which is the longest chord.

Activity 15

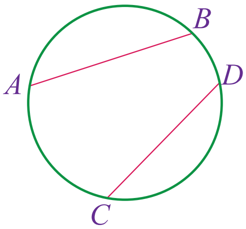
1. Find the length of a chord of a circle of radius 5 cm whose distance from the center of the circle is 4 cm.

2. $\overline{AB}$ and $\overline{CD}$ are two non - parallel chords of a circle. How do you locate the center of the circle using these chords.
3. What is the maximum number of intersection points of
   (a) a line and a circle (b) two circles

Exercises

Prove the following statements

1. Equal chords of a circle are equidistant from the center of the circle.
2. The perpendicular bisector of a chord of a circle passes through the center of the circle.
3. A radius is perpendicular to a tangent of a circle at the point of tangency.
4. Equal chords are subtended by equal arcs.

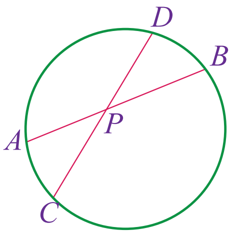


If $\overline{AB} \cong \overline{CD}$, then arc $AB \cong$ arc CD

5. If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary. Such a quadrilateral is said to be cyclic quadrilateral .

Circle Theorems(II)

If two chords of a circle intersect, then the product of the lengths of the segments are equal.



If $\overline{AB}$ and $\overline{CD}$ are chords of the circle intersecting at P, then $AP.\ BP = CP\ .\ DP$.

Example 17

In the figure, $AP = x - 2$, $AB = 7$, $CP = x - 3$ and $CD = 8$. Find x.

Solution

$BP = AB - AP$

$= 7 - (x - 2)$

$= 9 - x$

$DP = CD - CP$

$= 8 - (x - 3)$

$= 11 - x$

$AP \,.\, BP = CP \,.\, DP$

$\Rightarrow (x - 2)\,(9 - x) = (x - 3)\,(11 - x)$

$\Rightarrow -x^2 + 11x - 18 = -x^2 + 14x - 33$

$\Rightarrow 3x = 15$

$\Rightarrow x = 5.$

Example 18

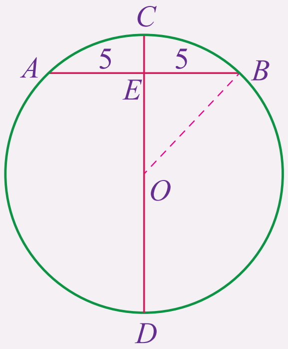
The lengths of two chords of a circle are 26 cm and 10 cm. If the longest chord bisects the shortest chord, find the lengths of the segments intercepted by the shortest chord.

Solution

Clearly the longest chord passes through the center of the circle as shown in the figure below.

$AB = BE = 5$ cm

Let $CE = x$, then $DE = 26 - x$



From the rectangle property of a circle,

$AE \,.\, BE = CE \,.\, DE$

$\Rightarrow 5 \times 5 = x(26 - x)$

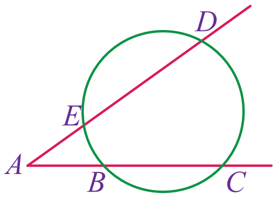
$\Rightarrow x^2 - 26x + 25 = 0$

$\Rightarrow (x - 1)\,(x - 25) = 0$

$\Rightarrow x = 1$ or $x = 25$

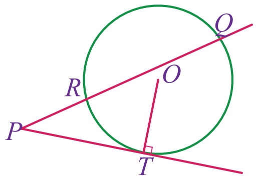
Hence $CE = 1$ cm and $DE = 25$ cm.

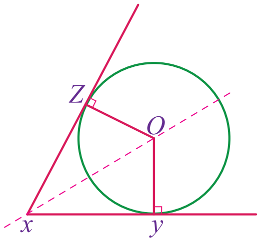
Secant Line and Tangent Line to a Circle



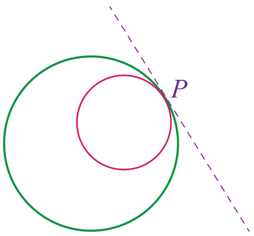
$\overleftrightarrow{AC}$ and $\overleftrightarrow{AD}$ are secant lines to the circle

$\overleftrightarrow{PT}$ is tangent to the circle and $\overleftrightarrow{PQ}$ is secant to the circle

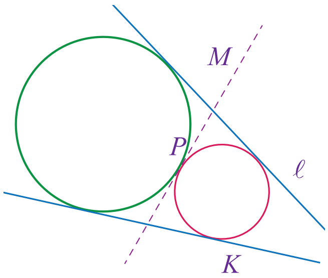




$\overleftrightarrow{XY}$ and $\overleftrightarrow{XZ}$ are tangent to the circle. The bisector of $\angle YXZ$ passes through the center of the circle and bisects the central angle, $\angle YOZ$.

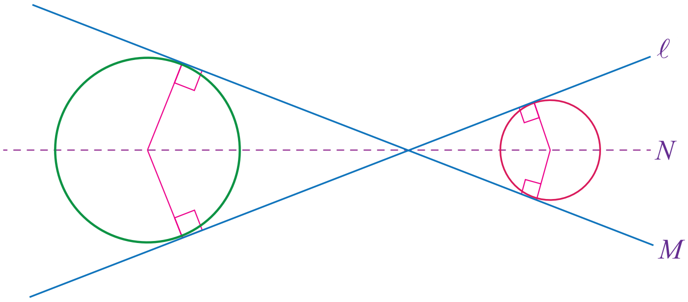


The line is tangent to both circles at P.



The lines ℓ and k are tangent to the circles at different points.

The line m is common tangent to both circles at the same point.



The lines ℓ and m are tangent to both circles at different points.

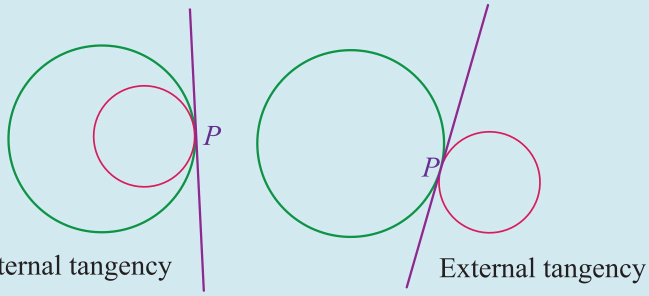
Line N bisects the central angles.

Tangent Circles

DEFINITION

Two circles are said to be tangent circles if they intersect at a single point. There are two types of tangency, internal and external.

In external tangency, the distance between the centers of the circles is equal to the sum of their radii.

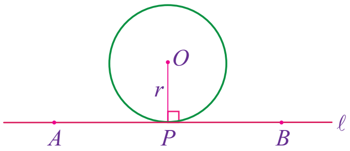


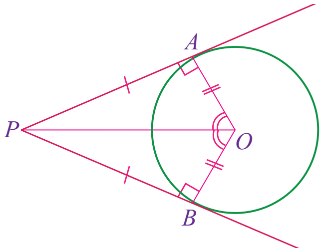
Properties of Tangents

1. The tangent to a circle is perpendicular to the radius of the circle at the point of contact.

   $\overleftrightarrow{AB}$ is tangent the circle at p.

   $\overline{OP} \perp \overleftrightarrow{AB}$



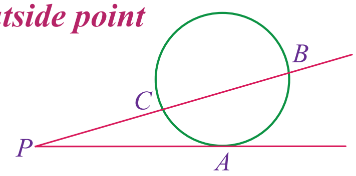


2. If two tangents are drawn from an external point of the circle, then they are congruent.

   $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to the circle.

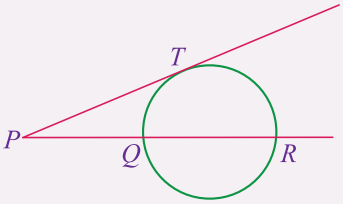
Secant and Tangent Lines drawn to a circle from an outside point

If $\overleftrightarrow{PA}$ is tangent to the circle at A and $\overleftrightarrow{PB}$ a secant line intersecting the circle at B and C, as shown in the figure, then $PA^2 = PC \,.\, PB$



Example 19

In the figure below $\overleftrightarrow{PT}$ is tangent to the circle at T and $\overleftrightarrow{PR}$ crosses the circle at Q and R. If $PQ = 5$ and $\overleftrightarrow{QR} = 15$, find PT.

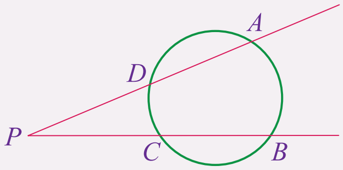


Solution

$$PT^2 = PQ \,.\, PR$$
$$= 5 \times (5 + 15)$$
$$= 5 \times 20$$
$$= 100$$
$$\Rightarrow PT = 10.$$

Example 20

If $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are secant lines intersecting the circle as shown below, then
$PC \,.\, PB = PD \,.\, PA$



Example 21

If in the above figure, $PD = 4$, $AD = 5$ and $PC = 3$, find CB.

Solution

$PD \,.\, PA = PC \,.\, PB$

$\Rightarrow 4 \times (4 + 5) = 3PB$

$\Rightarrow 3PB = 36$

$\Rightarrow PB = 12$

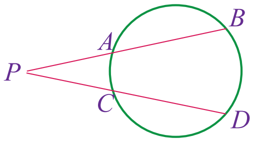
$\Rightarrow BC = PB - PC$

$= 12 - 3$

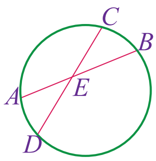
$= 9.$

Exercises

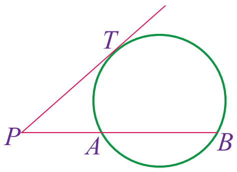
Find the unknown in the given figure.

1. 

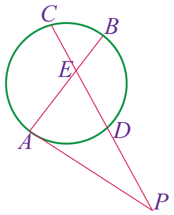
$PA = 4$, $AB = 5$, $PC = 3$, $CD = x$.

2. 

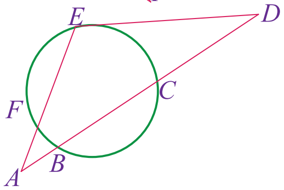
$AE = 3$, $BE = 8$, $CE = 4$, $DE = x$.

3. 

$\overline{PT}$ is tangent at T
$PA = 5$, $AB = 15$, $PT = x$.

4. 

$\overleftrightarrow{PA}$ is tangent at A
$AE = y$, $EB = x + 2$, $ED = 3$,
$EC = x$, $PD = 9$, $PA = 3x$.

5. 

$\overline{DE}$ is tangent at E.
$DE = 6$, $DC = 4$, $BC = y$, $AB = x$, $AF = 3$, $EF = 9$.

Key Terms

- Centroid
- Concurrent lines
- Circumcenter
- Cirucmcircle
- Convex polygon
- Concave polygon
- Exterior angle
- Equilateral triangle
- Equilateral polygon
- Equiangular polygon
- Interior angle
- Irregular polygon
- Isosceles triangle
- Incenter
- Incircle
- Orthocenter
- Parallelogram
- Plane
- Polygon
- Quadrilaterals
- Right triangle
- Regular polygon
- Rhombus
- Rectangle,
- Scalene triangle
- Secant line
- Segment
- Side
- Square
- Trapezium
- Tangent line
- Vertex

Summary

- **A plane** is a flat two dimensional surface that extends indefinitely.
- **A polygon** is a two dimensional geometric figure made of line segments connected to each other end to end.

- Types of polygons

  Polygons can be classified based on

  - The number of their sides as triangle, quadrilaterals
  - The measures of their interior angles as convex polygon and concave polygon.
  - The measures of their sides and angles as equilateral and equiangular polygons,
  - Their boundaries as simple and complex polygons.

- Regular polygon A polygon is a regular polygon if it has all sides congruent and all angles congruent.
- Interior and exterior angle sum of convex polygons

  - The sum of the measures of the interior angles of n - sided polygon is $(n-2) \times 180°$.
  - The measure of each interior angle of a regular polygon if n sides is $\frac{(n-2) \times 180^{\circ}}{n}$
  - The sum of the measures of the exterior angle of a polygon of n sides, taking one angle at each vertex is 360°.
  - The measure of each exterior angle of a regular polygon of n sides is $\frac{360^{\circ}}{n}$

- **Triangle**

  A triangle is a polygon that has three sides

  - Triangle can be classified based on their sides as scalene, isosceles and equilateral triangles.
  - Triangles can be classified by their angles as acute angle, right angle and obtuse angle triangle.

- Concurrent lines in a triangle

  - **Centroid:** The medians of a triangle intersect at a point called centroid.
  - **Incenter:** The angle bisectors of a triangle intersect at a point called in center.
  - **Circumcenter:** The perpendicular bisectors of sides of a triangle intersect at a point called circumcenter.

- **Orthocenter:** The altitude of a triangle intersect at a point called orthocenter.
- **Quadrilaterals:** A quadrilateral is a polygon that has four sides.
- **Trapezium:** A trapezium is a quadrilateral which has two and only two sides parallel.
- **Parallelogram:** A parallelogram is quadrilateral in which both pairs of opposite sides are parallel.
- Per meter formula perimeter of a regular polygon of n - sides and radius r is $n5$.

  The length of a side of a regular polygon is $s = 2r \sin \dfrac{180^o}{n}$.
- **Area formula:** Area of a regular polygon of n - sides, radius r and apolhom a is
  - $A = \dfrac{1}{2}ap$
  - $A = \dfrac{1}{2}nr^2 \sin \dfrac{360^o}{n}$
- **Tangent line:** A line is tangent to a circle if the line and the circle intersect at exactly one point called point of tangency.
- **Tangent circles:** Two circles are tangent to each other if they intersect exactly at one point.

Exercises

1. The measures of the angles of each interior angles of a convex hexagon are: x^o, $4x^o$, $4x^o$, $2x^o$, $3x^o$ and $2x^o$. Find x
2. The measure of an exterior angle of a regular polygon is 0.5^o, find the number of sides of the polygon.
3. The interior angles of a convex polygon add up to 8280^o. Find the number of sides of the polygon.
4. An exterior angle of a regular polygon measures 6^o. Find the measure of an interior angle of the polygon.
5. Find the measure of an interior angle of
   (a) a square
   (b) a regular pentagon
   (c) a regular dodecagon.

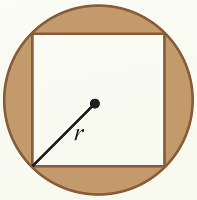
6. Find the measure of each interior angles of a regular polygon having
   (a) 10 sides
   (b) 15 sides
   (c) 24 sides
   (d) 36 sides
7. Find the number of sides of a polygon if the sum of the measures of the interior angles is
   (a) 2340^o
   (b) 3420^o
8. Find the number of sides of a regular polygon if the measure of each interior angle is
   (a) 160^o
   (b) 140^o
9. Find the area of a regular polygon having
   (a) radius 3 cm and 14 sides
   (b) radius 5 cm and 25 sides
10. An apothom of a regular octagon is 12 cm. Find its area.

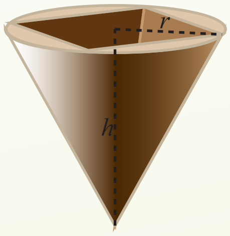
M12CH19

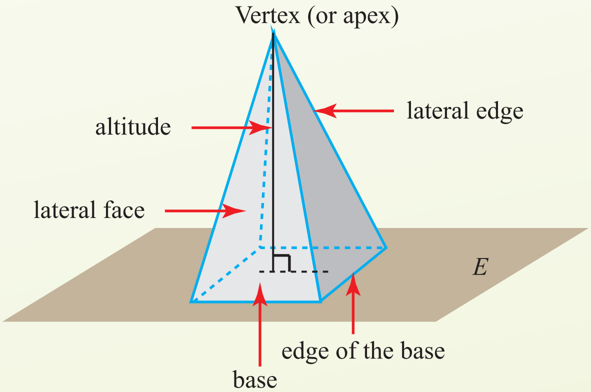
CHAPTER 19

SOLID GEOMETRY

Chapter Contents







Chapter Outcomes

Upon completion of this chapter, learners will:

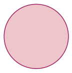
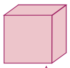
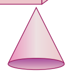
- identify and discuss common solids, face of a solid;
- define and discuss the following prism, cuboid, cylinder, pyramids, tetrahedron, hexagonal pyramid, cone;
- measure the lengths and angles in solids;
- calculate volumes and surface area.

Introduction

Solid geometry is a branch of geometry that deals with three dimensional space. The objects around us have different dimensions.

In this unit we will define geometric figures and then calculate the surface areas and volume of some solid figures such as prism, cylinder, cone and pyramid.

Let us consider the following figures and identify their dimensions in order to learn about geometric solid figures.

1. Point .
2. Line segment ______
3. Can 
4. Rectangle 
5. Circle 
6. Basketball 
7. Box 
8. Cone 

In the above mentioned figures:

1. A point has no size, hence it has no dimension.
2. A line segment and a line have no width but have length, hence they are one dimensional.
3. A circle and rectangle, have width and length but have no thickness, hence they are two dimensional.
4. A box, a cone and basketball have length, width and height and hence they are three-dimensional.

In one-dimensional figures we had been calculating length, in two-dimensional figures we had been calculating lengths of sides perimeters, and areas. In three-dimensional figures we are calculating lengths, areas and volumes.

Solid geometry deals with three dimensional objects. A geometric solid has length, width and height which is defined as follows.

Definition

A geometric solid is a region of spaces enclosed by planes and curved surfaces. Such as prisms, pyramids, spheres, cylinders, and cones. If the solid is enclosed by polygons it is called polyhedrons. The polygons are called faces. The segment which is the intersection point of any two faces of a polyhedron is called an edge. The intersection point of three or more edges is called vertex.

Definition

A regular polyhedron or platonic solid is a polyhedron whose faces are congruent regular polygons and which have the same number of faces intersecting at each vertex.

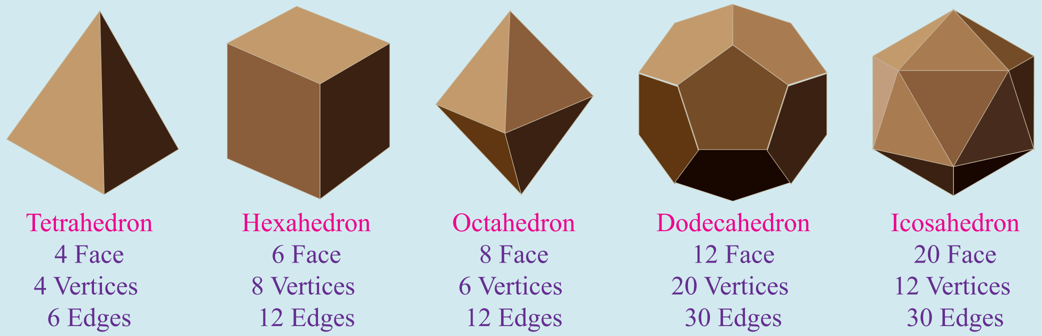


Figure 1.

There are only five regular polyhedrons. Tetrahedron, hexahedron, octahedron dodecahedron, icosahedron

The faces of regular polyhedrons are equilateral triangles, squares, or regular pentagons.

Prisms

A prism is a three dimensional geometric figure with two congruent polygons called bases on parallel planes and all line segments joining the corresponding points of the bases. Figure 2 is a prism with

Vertices, *A, B, C, D, E, F, G, H, I, J, K, L, M* and *R*

- The polygons *ABCDEFG* and *IJKLMNR* are called bases; lower and upper bases.

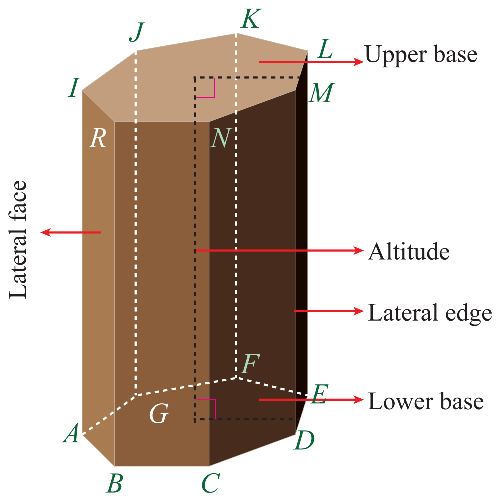


Figure 2.

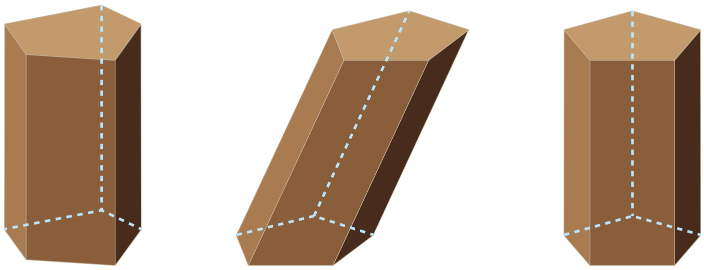
- The quadrilaterals *ABRI*, *BCNR*, *CDMN*, *DELM*, *EFKL*, *FGJK*, *AGJI* are called lateral faces.
- The line segments $\overline{AI}, \overline{BR}, \overline{CN}, \overline{DM}, \overline{EL}, \overline{FK}$ and $\overline{GJ}$ are said to be lateral edges.

In a prisms, the sides of the base are said to be edges of the base. The lateral edges and edges of the base are said to be edges of the prism. The height of the prism is the distance between the planes of the bases.

Prisms are classified as right prisms and oblique prisms.

A right prism is a prism whose lateral edges are perpendicular to the bases.

An oblique prisms is a prism whose lateral edges are not perpendicular to the bases.



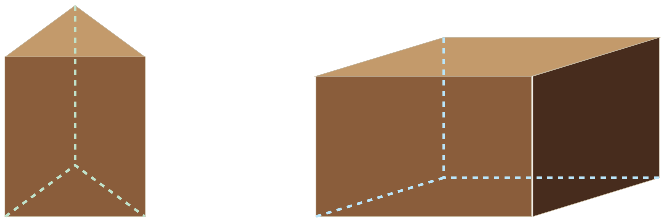
Right prism Oblique prism Regular prism

Figure 3.

The lateral faces of a prism are a parallelogram. The lateral faces of a right prism are rectangle.

In this book we consider only right prisms. Prisms are also classified according to their bases.

- **Triangular prism:** A prism whose bases are triangles.
- **Cuboid or Rectangular prism:** A prism whose bases are rectangles.
- All angles of a cuboid measure 90^o.
- **Parallelepiped:** A prism whole bases are parallelograms.



Triangular prism Parallelepiped

Figure 4.

- **Pentagonal prism:** A prism whose bases are pentagons.
- **Cube:** A prism whose bases and whose lateral faces are congruent squares.
- **Regular prism:** A right prism whose bases are regular polygons.

Exercises

Fill in the blank

1. A right square prism with congruent faces is called ______________.
2. If the base of a prism is a triangle, then it is called ______________.
3. If the base of a prism is a parallelogram, then it is called ______________.

Cylinder

Activity 1

What is the shape of the prism if the bases are replaced by circular regions?

Definition

A cylidner is a three dimensional geometric figure composed of two congruent circular regions called bases on parallel planes and all line segments parallel to the line segment joining the centers of the bases with end points on the bases.

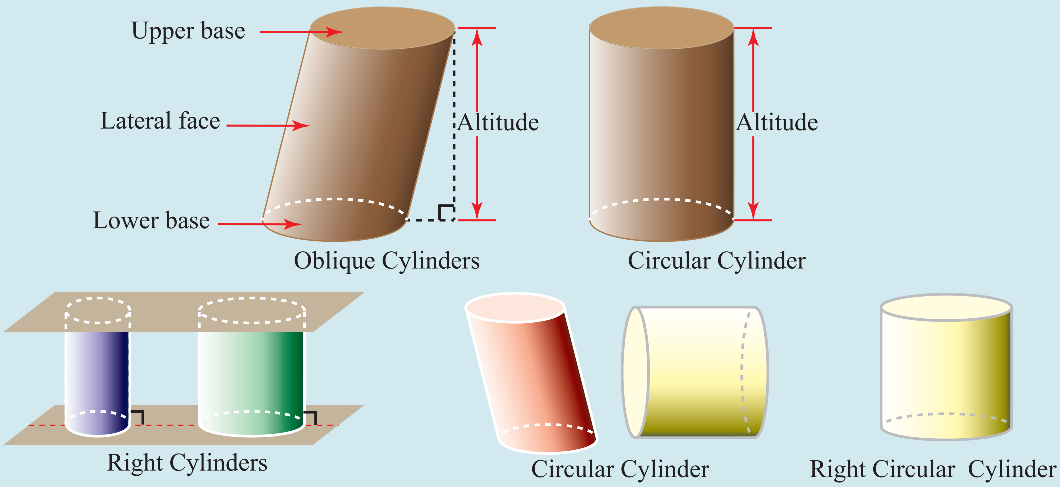


Figure 5.

Activity 2

1. What is the volume of a cube whose edge is 1cm?
2. If two solids are congruent, verify that they have equal volume
3. If a solid is divided into two, then verify that the volume of the original solid is the sum of the volumes of the two pieces.
4. Find the volume and surface area of a cuboid or a rectangular prism whose dimensions are *l*, *w*, and *h*.

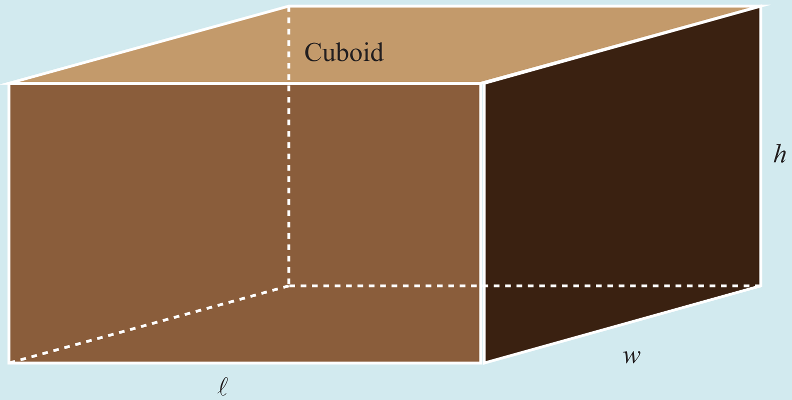


Figure 6.

Theorem: The volume of a prism is equal to the product of the area of its base and its altitude. $V = A_B \times h$; where A_B is the area of the base and h is its altitude.

Example 1

In a triangular prism, let the lengths of sides of its bases be 3 cm, 4 cm, and 5 cm. Find the volume of the prism if it has an altitude of 6 cm.

Solution

The area of the base of the prism is the area of a right triangle whose legs are 3 cm and 4 cm.

Which is $\frac{1}{2} \times 3 \times 4 \text{ cm}^2 = 6.$

$\therefore V = A_B \times h = 6 \times 6 \text{ cm}^3 = 36 \text{ cm}^3$

Exercises

Find volume of the following:

1. 

Figure 7.

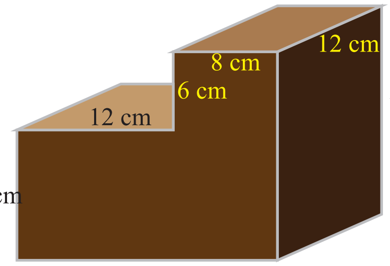
2. 

Figure 8.

The Volume and Surface Area of Prisms and Cylinders

The Volume and Surface area of a Prism

Base area: The area of the base of a prism is the area of the polygon which is the base of the prism and it is represented by A_B.

Lateral Surface area: The lateral surface area of a prism represented by A_L is the sum of the areas of the lateral faces.

If P is the perimeter of the base of a right prism of height h, then $A_L = Ph$.

The total surface area of a right prism which is represented by A_T is the sum of the areas of the bases and the lateral area of the prism:

$$A_T = 2A_B + A_L.$$

The Volume V of a prism of altitude h and area of the base A_B V is:

$$V = A_B h.$$

Example 2

In a cuboid, if the dimensions of the base are 6 cm and 5 cm and the length of the altitude is 2 cm, find

(a) the total surface area

(b) the volume.

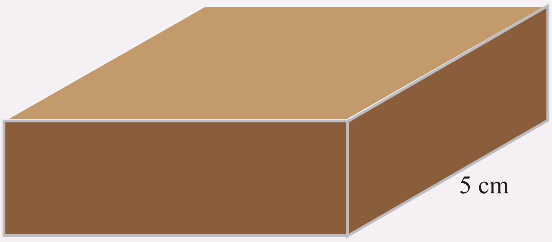


Figure 9.

Solution

$A_B = 6 \times 5 \text{ cm}^2 = 30 \text{ cm}^2$

$P = 2\,(6 + 5)$ cm, the perimeter of the base = 22 cm.

$A_L = Ph = 22 \text{ cm} \times 2 \text{ cm} = 44 \text{ cm}^2$

$A_T = A_L + 2A_B = 44 \text{ cm}^2 + 2\,(30) \text{ cm}^2 = 104 \text{ cm}^2$

$V = A_B h = 30 \text{ cm}^2 \times 2 \text{ cm} = 60 \text{ cm}^3$.

EXAMPLE 3

Express the total surface area and volume of a cuboid whose dimensions are a, b, and c. Find the length of the diagonal $\overline{AG}$ if $a = 6$ cm, $b = 8$ cm and $c = 24$ cm.

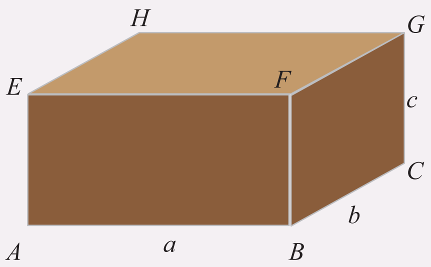


Figure 10.

Solution

$A_B = ab$

$P = 2\,(a + b)$

$A_L = Ph = 2\,(a + b)c$

$A_T = 2A_B + A_L = 2ab + 2\,(a + b)c$

$= 2ab + 2ac + 2bc$

$= 2\,(ab + ac + bc)$

$V = abc$

$AG^2 = AC^2 + CG^2$

$\Rightarrow AG^2 = AB^2 + BC^2 + CG^2$

$\Rightarrow AG^2 = (6^2 + 8^2 + 24^2) \text{ cm}^2$

$= 26^2 \text{ cm}^2$

$\Rightarrow AG = 26$ cm.

Example 4

If the base of a right prism is a regular hexagons of perimeter 30 cm and the altitude of the prism is equal to 10 cm, find

(a) the total surface area.

(b) the volume of the prism.

Solution

The area A of a regular polygon of radius r and number of sides n is.

$$A = \frac{1}{2}nr^2\sin\frac{360^\circ}{n}$$

In a regular hexagon, $P = 6s$ and $r = s \Rightarrow$ 30 cm = $6r \Rightarrow r = 5$ cm

$$\therefore A_B = \frac{1}{2}\times 6\times(5\text{cm})^2\sin\frac{360^\circ}{6} = 75\text{ cm}^2\left(\frac{\sqrt{3}}{2}\right) = \frac{75\sqrt{3}\text{ cm}^2}{2}$$

$$A_L = Ph = 30\text{ cm} \times 10\text{ cm} = 300\text{ cm}^2$$

$$A_T = 2A_B + A_L = 2\times\frac{75\sqrt{3}\text{ cm}^2}{2} + 300\text{ cm}^2 = \left(75\sqrt{3}+300\right)\text{cm}^2$$

$$V = A_B\,h = \frac{75\sqrt{3}}{2}\times 10\text{ cm}^3 = 375\sqrt{3}\text{ cm}^3.$$

Example 5

The base of a right prism is a rhombus whose diagonals are 8 cm and 6 cm long. The altitude of the prism is 12 cm. Find the volume of the prism.

Solution

The area of a rhombus with diagonals d_1 and d_2 is

$A_B = \frac{1}{2}d_1 d_2$, where $d_1 = 6$ cm, $d_2 = 8$cm

$$= \frac{1}{2}.6\text{ cm} . 8\text{ cm} = 24\text{ cm}^2 = \frac{1}{2}. 6\text{ cm} . 8\text{ cm} = 24\text{ cm}^2$$

$$\Rightarrow V = A_B \times h \Rightarrow V = 24\text{ cm}^2 \times 12\text{ cm} \Rightarrow V = 288\text{ cm}^3.$$

Example 6

If the diagonal of a face of a cube is s unit, find the volume of the cube in terms of s.

Solution

Let the lengths of sides of the bases of the cube be x, then

$$s^2 = x^2 + x^2 \text{ or } 2x^2 = s^2 \text{ or } x = \frac{\sqrt{2}}{2}s$$

$$V = x^3 = \left(\frac{\sqrt{2}}{2}s\right)^3 = s^3\frac{\sqrt{2}}{4}.$$

The Volume and Surface Area of Cylinders

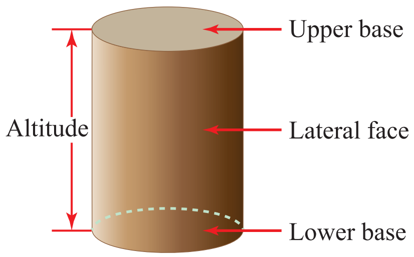
C = circumference = $2\pi r$

$A_B = \pi r^2$

$A_L = Ch = 2\pi rh$

$A_T = 2A_B + A_L = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$

$V = A_B h = \pi r^2 h$.



Right circular cylinder

Figure 11.

Example 7

Find the volume and total surface area of a right circular cylinder with altitude 5cm and radius of the base 3 cm.

Solution

$r = 3$ cm; $h = 5$ cm

$C = 2\pi r = 6\pi$ cm

$A_B = \pi r^2 = \pi(3 \text{ cm})^2 = 9\pi \text{ cm}^2$

$A_L = Ch = 6\pi \text{ cm} \times 5 \text{ cm} = 30\pi \text{ cm}^2$

$A_T = 2A_B + A_L = 2(9\pi) \text{ cm}^2 + 30\pi \text{ cm}^2 = 48\pi \text{ cm}^2$

$V = \pi r^2 h = \pi(3 \text{ cm})^2 \times 5 \text{ cm} = 45\pi \text{ cm}^3$.

Example 8

In a right circular cylinder of altitude 21 cm the lateral area is $84\pi \text{ cm}^2$, find the volume of the cylinder.

Solution

$A_L = 84\pi \text{ cm}^2$

$\Rightarrow 2\pi rh = 84\pi \text{ cm}^2$

$\Rightarrow 2\pi r \times 21 \text{ cm} = 84\pi \text{ cm}^2$

$\Rightarrow r = 2 \text{ cm}$

$V = \pi r^2 h = \pi(2)^2 \times 21 \text{ cm}^3 = 84\pi \text{ cm}^3$.

Example 9

The total surface area of a right circular cylinder is 48π cm^2. If the height of the cylinder is 5cm, find the radius of the lower base of the cylinder.

Solution

Given: $2\pi rh + 2\pi r^2 = 48\pi$ cm^2 and $h = 5$ cm.

Hence $2\pi r \times 5 + 2\pi r^2 = 48\pi$

$\Rightarrow 5r + r^2 = 24$

$\Rightarrow r^2 + 5r - 24 = 0$

$\Rightarrow (r + 8)(r - 3) = 0$

$\Rightarrow r = -8$ or $r = 3$.

Therefore, the radius of the cylinder is 3cm since radius is never negative.

Example 10

A square prism is inscribed in a cylinder of height h and radius of the base r. Find the ratio of the volume of the cylinder to the volume of the prism.

Solution

The prism and the cylinder have equal height h. The area of the base of the prism is $2r^2$.

Therefore, the volume of the cylinder is $\pi r^2 h$ and of the prism is $(2r^2)\,h$.

Therefore, the ratio of the volume of the cylinder to the volume of the prism is $\frac{\pi r^2 h}{2r^2 h} = \frac{\pi}{2}$.

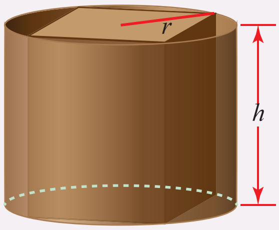


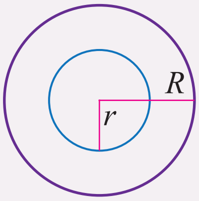
Figure 12.

Example 11

The thickness of aright cylindrical shell is 6 cm. The outer diameter of the shell is 32 cm. If the height of the shell is 80 cm, find the volume and the external surface area of the shell.

Solution

Consider the bases of the shell.



The thickness of the shell is R - r where $R = \frac{1}{2}(32)\text{cm}$.

Thus, $16 - r = 6$

$\Rightarrow r = 10$ cm.

The volume of the shell is the difference between the volumes of the cylinders whose radii of bases are R and r which is

$\pi R^2h - \pi r^2h = \pi h(R^2 - r^2)$.

Here $R = 16$ cm, $r = 10$ cm and $h = 80$ cm

Thus, the volume of the shell is

$T \times 80\ (16^2 - 10^2)\ \text{cm}^3$

$= 1280\pi\ \text{cm}^3$.

When the shell is viewed from outside, the bases have holes.

Thus, the external surface area of the shell is $2\pi Rh + 2\pi R^2 - 2\pi r^2$

$(2\pi \times 16 \times 10 + 2\pi \times 16^2 - 2\pi \times 10^2)\ \text{cm}^2$

$= (320\pi + 512\pi - 200\pi)\ \text{cm}^2$

$= 632\pi\ \text{cm}^2$.

Exercises

1. The dimensions of a cuboid are 15 cm, 9 cm and 4 cm. Find the surface area and volume of the cuboid.
2. A rectangular bath is 8m long, 6 m wide and 3 m deep. How much deeper must it be made to hold 48 m^3 of water more?
3. A right rectangular prism and cube have the same volume. Find the edges of the cube if the edges of the rectangular prism are 6 m, 7 m and 12 m.
4. Find the cost of painting the internal and external walls of a house whose rectangular dimensions are 17 m, 16 m and height 6 m if the cost of painting of 1 m^2 is L$ 5.

5. Find the volume and surface area of a right prism of height 10 cm and whose base is
   (a) an equilateral triangle of side 3 cm.
   (b) a square of side 3 cm.
   (c) an isosceles trapezium with bases 4 cm and 5 cm and height 6cm
   (d) a regular hexagon with radius 6 cm.
6. Find the volume and total surface area of a right circular cylinder if the area of the base is 100π cm^2 and its altitude is 14 cm.
7. Find the volume and total surface area of a right circular cylinder with base radius 5 cm and height 6 cm.
8. The circumference of the base of a right circular cylinder is 20π cm. If the altitude of the cylinder is 12 cm, find its volume and total surface area.
9. A manufacture makes two different types of closed right circular cans. The first type has altitude 10 cm and radius of the base 5 cm. The second type has altitude 5 cm and radius of the base 10 cm. It is required to pack 25000π cm^{3}material in each type. Which type requires more metal to make?
10. How many cans are required from each type?
11. A cube is inscribed in a right circular cylinder of radius of the base r. A second cube is circumscribed in the same cylinder. Find the ratio of the total surface area of the larger cube to the smaller cube.
12. The volume of a cube is 0.216 c.c. Find its total surface area.
13. Find the ratio of the lateral surface area to the total surface area of a cube.

A circular cone is a three dimensional figure consisting of all line segments joining a single point called vertex or apex to every point of a circular region called base which is not on the plane of the apex.

Cone

Cones are classified as right ciruclar cone or oblique cone depending upon the position of the apex on the base.

- The **Lateral surface is** the surface of the cone between V and E that doesn't include the base. It is the curved conical surface.

Activity 3

What type of solid is generated if you rotate the following right triangle *VOA* about its leg $\overline{OV}$?

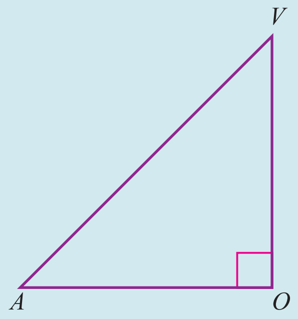
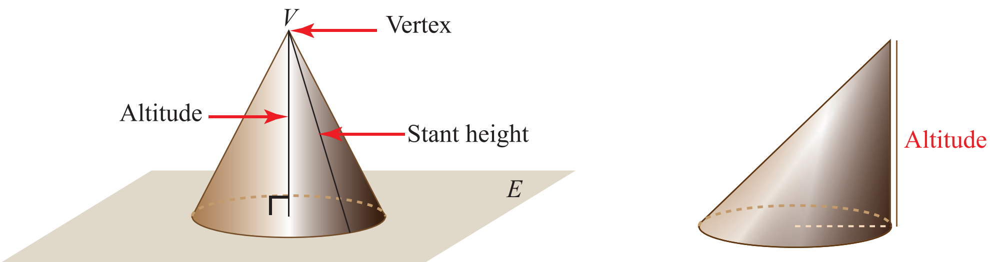


Figure 13.

- The altitude of the cone is the perpendicular distance between the vertex and the plane of the base.
- A cone is a **right circular** cone if the altitude of the cone joins the apex and the centre of the cone other wise it is an **oblique circular cone**.
- **Slant height:** The distance from the vertex to the circumference of the base of a right circular cone is said to be slant height.



Right circular cone

Oblique circular cone

Figure 14.

Pyramid

Activity 4

What type of solid is formed if the base of a cone is replaced by a polygonal region?

DEFINITION

A pyramid is a three dimensional figure consisting of all line segments joing a single point called apex or vertex to every point of a polygonal base which is not on the plane of the base. The lateral faces of a pyramid are triangular regions.

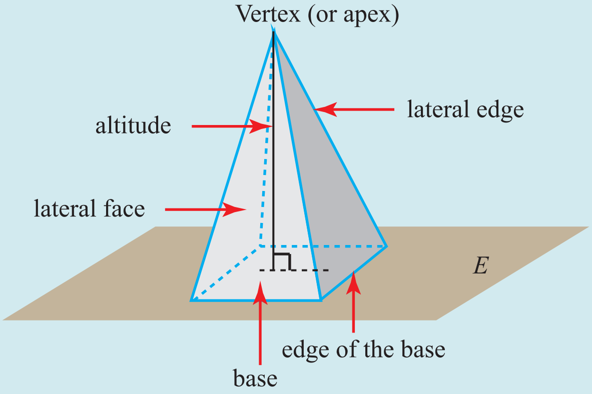


Figure 15.

DEFINITION

If the base of a pyramid is a triangular region, then it is called **tetrahedron.**

Regular pyramid: If the base of a pyramid is regular polygon and if the apex of the pyramid is equidistant from each vertex of the pyramid, then it is called regular pyramid. *The lateral faces of a regular pyramid are congruent isosceles triangles*.

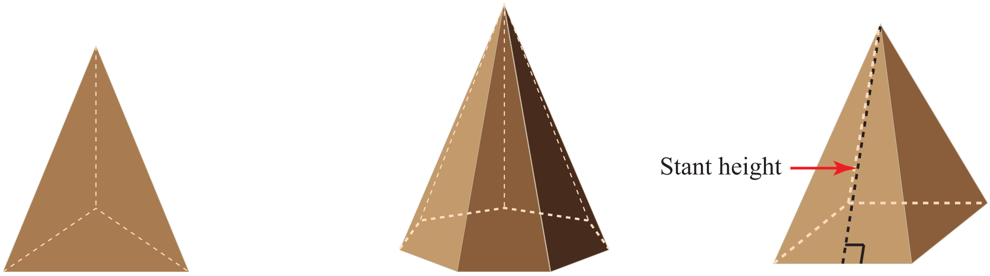


Figure 16.

The slant height of regular pyramid is the perpendicular distance from the apex to any side of the base. The altitude of a pyramid is the distance from the apex to the plane of the base.

If the base of a pyramid is quadrilateral, then it is called quadrangular pyramid. If the base of a pyramid is hexagon, then it is called hexagonal pyramid.

The Volume and Surface Area of a Right Pyramid

The formula for the volume of a pyramid a one third of the product of the area of its base and its altitude. It is similar to the volume of a cylinder. The following figure explanes this relation ship.

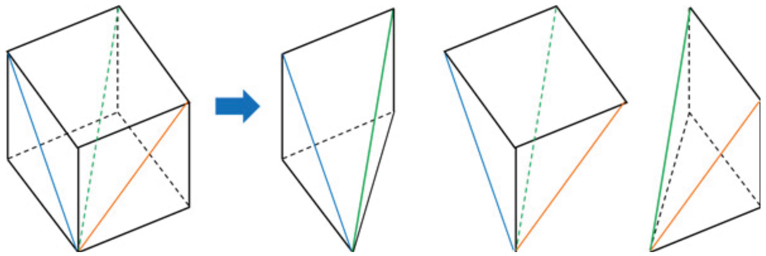


Figure 17.

A_B = The area of the base

A_L = The area of the triangular lateral faces = $\frac{1}{2}p\ell$, ℓ is the slant height

$$A_T = A_B + A_L.$$

$$V = \frac{1}{3}A_B h.$$

Example 12

Find the volume of a pyramid of altitude 10 cm and area of the base 33 cm^2.

Solution

$V = \frac{1}{3}A_B h$; $A_B = 33$ cm^2 and $h = 10$ cm

$$= \frac{1}{3}\times 33 \times 10 \text{ cm}^3 = 110 \text{ cm}^3.$$

Example 13

The lateral area of a pyramid is 45% of its total surface area. If the pyramid is a regular square pyramid, find the ratio of the length of the base to the slant height.

Solution

Let ℓ be the slant height and s be the length of a side of the base.

$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(4s)\ell = 2s\ell$$

It is given that $\frac{A_L}{A_T} = \frac{45}{100} = \frac{9}{20}$ where $A_T = A_B + A_L$

$$\frac{2s\ell}{2s\ell + s^2} = \frac{9}{20}$$

$$40s\ell = 18s\ell + 9s^2$$

$$22s\ell = 9s^2$$

$$s = \frac{22\ell}{9}$$

$$\frac{s}{\ell} = \frac{22}{9}.$$

Therefore, the ratio s : ℓ = 22:9

Example 14

A cube made of metal is melted down and formed into a regular square pyramid whose base is the same as the base of the cube. Find the ratio of the total surface area of the pyramid to the total surface area of the cube.

Solution

Let the lengths of the sides of the cube be s, and h is the altitude of the pyramid. The volume of the cube = the volume of the pyramid.

$$\Rightarrow s^3 = \frac{1}{3}s^2h$$

$$\Rightarrow h = 3s$$

$$\ell^2 = h^2 + \left(\frac{s}{2}\right)^2 = (3s)^2 + \frac{s^2}{4} = \frac{37s^2}{4}$$

$$\Rightarrow \ell = \frac{s}{2}\sqrt{37}.$$

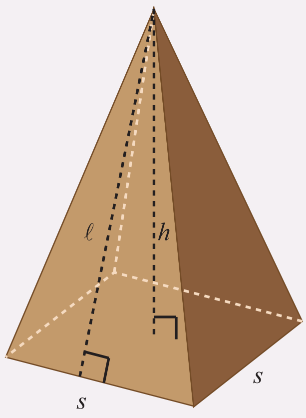


Figure 18 .

The total surface area of the cube = $6s^2$.

The total surface area of the pyramid = $s^2 + \frac{1}{2}P\ell = s^2 + \frac{4(s)\ell}{2} = s^2 + 2sl.$

$\Rightarrow s^2 + 2sl = s^2 + 2s\ \frac{s}{2}\sqrt{37} = s^2\left(1+\sqrt{37}\right).$

But $\frac{s^2(1+\sqrt{37})}{6s^2} = \frac{1+\sqrt{37}}{6}$ .The required ratio is $\left(1+\sqrt{37}\right)$: 6.

Example 15

The length of each edge of a regular square pyramid is 4 cm. Find the volume and surface area of the pyramid.

Solution

Let the altitude of the pyramid be h.

Let the diagonals of the base $ABCD$ meet at E.

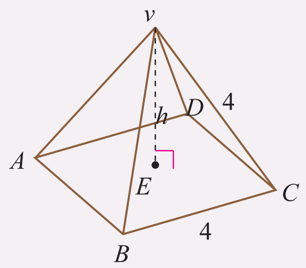


Figure 19 .

Then $EC = \frac{1}{2}AC = \frac{1}{2}\sqrt{AB^2 + BC^2}$, from Pythagoras.

$= \frac{1}{2}\sqrt{4^2 + 4^2}$

$= \frac{1}{2}\times 4\sqrt{2} = 2\sqrt{2}$.

$\Delta\ VCE$ is right angle at $\angle E \Rightarrow h^2 + \left(2\sqrt{2}\right)^2 = 4^2 \Rightarrow h^2 = 16 - 8 = 8.$

$\therefore\ \ h = 2\sqrt{2}$

(i) $V = \frac{1}{3}A_B h$

$= \frac{1}{3}.4^2.2\sqrt{2}\text{cm}^3$

$= \frac{32\sqrt{2}}{3}\ \text{cm}^3.$

(ii) $A_B = (4\text{cm})^2 = 16\text{cm}^2$

$A_L = 4\times\frac{4^2\sqrt{3}}{4}\text{cm}^2 = 16\sqrt{3}\text{cm}^2$

$A_T = 16\ \text{cm}^2 + 16\sqrt{3}\text{cm}^2$

$= 16\ (1 + \sqrt{3}\ )\ \text{cm}^2.$

Example 16

Find the volume and surface area of a regular tetrahedron whose edges measure y units.

Solution

Each of the six edges of the regular tetrahedron is equal to y as shown below.

In a regular polygon the length of the side,

$s = 2r \sin \frac{180^0}{n}$

$$\Rightarrow y = 2r\sin\frac{180^o}{3} = 2r\sin 60^0$$

$$= 2r\left(\frac{\sqrt{3}}{2}\right) = r\sqrt{3}$$

$$\Rightarrow r = \frac{y}{\sqrt{3}} = \frac{y\sqrt{3}}{3}.$$

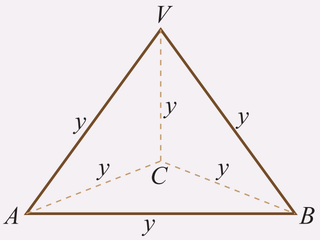


Figure 20.

Then, $h^2 + r^2 = y^2$

$$\Rightarrow h^2 = y^2 - r^2 = y^2 - \left(\frac{y}{3}\sqrt{3}\right)^2$$

$$= y^2 - \frac{y^2}{3} = \frac{2}{3}y^2$$

$$\Rightarrow h = y\sqrt{\frac{2}{3}} = \frac{y\sqrt{6}}{3}.$$

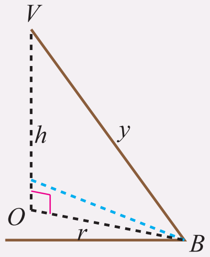


Figure 21.

(a) $A_B = \dfrac{y^2\sqrt{3}}{4}$, the area of an equilateral triangle

$$V = \frac{1}{3}A_B h = \frac{1}{3}\times\frac{y^2\sqrt{3}}{4}\times\frac{y\sqrt{6}}{3} = \frac{1}{3}\frac{y^3\times 3\sqrt{2}}{12} = \frac{y^3\sqrt{2}}{12}.$$

(b) $A_L = 3\left(\dfrac{y^2\sqrt{3}}{4}\right) = \dfrac{3y^2\sqrt{3}}{4}$

$$A_T = A_B + A_L = \frac{y^2\sqrt{3}}{4} + \frac{3y^2\sqrt{3}}{4} = y^2\sqrt{3}.$$

Example 17

The base of a regular pyramid is a square whose sides measure 4cm each. If the lateral edges measure 5cm each, find its surface area.

Solution

(a) $A_B = 4^2 = 16$

(b) One of the lateral face is an isosceles triangle with sides 5, 5 and 4.

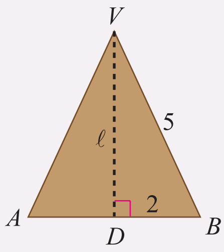


Figure 22.

Therefore, in $\triangle AVB$, $AD^2 + VD^2 = AV^2$; where $AD = BD \Rightarrow \ell^2 = 25 - 4 \Rightarrow$

$\ell = \sqrt{21}$

Therefore, the slant height is $\sqrt{21}$ and the area of one lateral face is

$\frac{1}{2} \times \sqrt{21} \times 4 = 2\sqrt{21}$.

$\Rightarrow$ The lateral area $= 4(2\sqrt{21}) = 8\sqrt{21}$

$\therefore$ Surface area $= A_T = A_L + A_B = 16 + 8\sqrt{21}$.

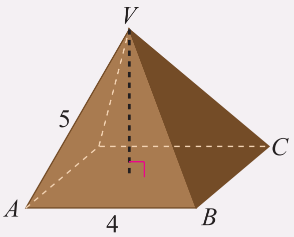


Figure 23.

Exercises

I True or False Questions.

1. Write true if the statement is correct otherwise write false.
2. The base of a regular pyramid must be a square.
3. The altitude of a regular pyramid must be perpendicular to the base and pass through the center of the base.
4. The total surface area of a regular triangular pyramid, all of whose six edges are s units, is $s^2\sqrt{3}$.
5. If in a square pyramid the edge of the base is doubled and the height of the pyramid is halved, then the volume is doubled.
6. The side of the square base of a pyramid is 15 and its height is 5. Then, there is no difference in new volume whether the base is doubled or the height is increased by 15.

II. Work Out

Solve the following problems step by step.

7. Find the volume and total surface area of a regular square pyramid if its slant height is 13 cm and the edge of the base is 10 cm.
8. Find the lateral area of regular square pyramid of volume $\frac{2}{3}x^3$ and altitude $2x$.
9. An equilateral triangle of side s is rotated in space about one of its side. Find the volume of the solid.
10. A solid metallic cylinder with radius 6 cm and height 18 cm is melted down and recast as a right circular solid cone with radius 9. Find the height of the cone.

11. Find the lateral area of a regular square pyramid if the edge of the base is x and its slant height is y.
12. In a regular square pyramid each of the lateral face is an equilateral triangle of side x. Find
    (a) the slant height
    (b) the volume
    (c) total surface area

The Volume and Surface Area of a Right Circular Cone

$$A_B = \pi r^2$$

$$V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$$

$$C = 2\pi r \text{ and } \ell = \sqrt{r^2 + h^2}$$

$$A_L = \frac{1}{2}C\ell = \frac{1}{2}(2\pi r)\ell = \pi r\ell$$

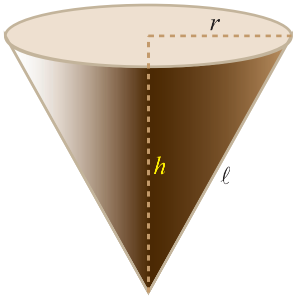


Figure 24.

Example 18

A right circular cone has radius of the base 5 cm and height 12 cm. Find
(a) the volume
(b) the total surface area of the cone.

Solution

$r = 5$ cm and $h = 12$ cm

$$\Rightarrow \ell = \sqrt{5^2 + 12^2}\text{cm} = 13 \text{ cm}$$

$$A_B = \pi r^2 = \pi(5\text{cm})^2 = 25\pi \text{ cm}^2$$

$$V = \frac{1}{3}A_B h = \frac{1}{3} \times 25\pi \times 12\text{cm}^2 = 100\pi\text{cm}^2$$

$$A_L = \frac{1}{2}C\ell = \pi r\ell = \pi \times 5 \times 13\text{cm}^2 = 65\pi\text{cm}^2$$

$$A_T = A_B + A_L = (25\pi + 65\pi)\text{ cm}^2 = 90\pi \text{ cm}^2.$$

Example 19

A right circular conical solid of height 4cm and radius of the base 3 cm is dropped into a right circular cylinder of height 8 cm and radius of the base 4 cm which is half filled with water and sinks. How much does the water rise?

Solution

The volume of the water displaced is equal to the volume of the conical solid.

The volume of the displaced water = $\frac{1}{3}\pi(3\text{ cm})^2 \times 4\text{ cm} = 12\pi\text{ cm}^3$

$\pi r^2 h = 12\pi\text{ cm}^3$, (*the volume of the displaced water in the cylinder*).

$\pi(4\text{ cm})^2 h = 12\pi\text{ cm}^3$

$h = \frac{12}{16}\text{cm} = 0.75\text{ cm}$

$\therefore$ The water rises by 0.75 cm.

Example 20

A right square pyramid is inscribed in a right circular cone of radius of the base r and height h as shown in the figure below. If the pyramid is removed, find the volume and total surface area of the remaining part.

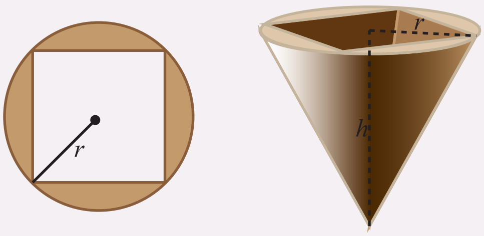


Figure 25.

Solution

Consider the base of the solid.

The area of the shaded part is equal to

$$\pi r^2 - \frac{1}{2} \times 4 \times r^2 \sin\frac{360^0}{4} = \pi r^2 - 2r^2$$

V = the volume of the cone – the volume of the pyramid

$$= \frac{1}{3}\pi r^2 h - \frac{1}{3}(2r^2 h) = \frac{1}{3}r^2 h(\pi - 2)$$

$$A_T = (\pi r^2 - 2r^2) + \pi r\ell + 2r\ell\sqrt{2}$$

$$= (\text{base}) + A_L(\text{the cone}) + A_L(\text{the pyramid})$$

Example 21

A right circular cylinder is drilled in the form of a right circular cone. If the volume of the remaining solid is 192π and its altitude is 8, find the total surface area of the solid.

Solution

Let r be the radius of the base and h the altitude of the cylinder. The volume of the solid is equal to the volume of the cylinder minus the volume of the cone.

$$V = \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$$

$$\frac{2}{3}\pi r^2 h = 192\pi \text{ and } h = 8$$

$$\frac{2r^2}{3}\times 8 = 192 \Rightarrow r^2 = \frac{192}{8}\times\frac{3}{2} = 36$$

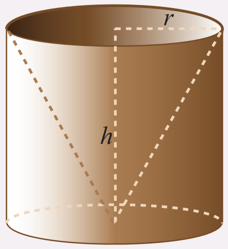


Figure 26.

Therefore, $r = 6$.

The total surface area of the solid is equal to the sum of the lateral areas of the cone and the cylinder and the area of the lower base of the cylinder.

$$\Rightarrow A_T = \pi r^2 + \pi r\ell + 2\pi rh = \pi\times 6^2 + \pi\times 6\times 10 + 2\pi\times 6\times 8 = 192\pi.$$

Example 22

A sector of radius 5cm and measure of central angle 216^0 is folded to make a cone. Find the volume of the cone.

Solution

The arc length of the sector is equal to the circumference of the base of the cone.

$\Rightarrow \frac{\pi\ell\theta}{180} = 2\pi r$, where ℓ is the radius of the sector which is going to be the slant height of the cone.

$$\Rightarrow \frac{\pi\times 5\text{cm}\times 216}{180} = 2\pi r$$

$$\Rightarrow r = \frac{\pi\times 5\ \times 216\text{cm}}{180\times 2\pi} = 3 \text{ cm}$$

From Pythagoras, $h^2 + r^2 = 5$

$$\Rightarrow h^2 = 5^2 - r^2 = 5^2 - 3^2 = 16$$

$$\Rightarrow h = 4$$

$$\Rightarrow V = \frac{1}{3}\pi(3)^2\times 4\text{cm}^3 = 12\pi\text{cm}^3.$$

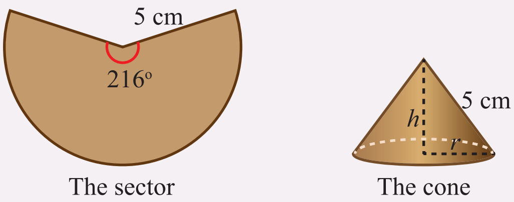


Figure 27.

EXERCISES

1. Find the volume and total surface area of a cone of height 5 cm and diameter of the base 24 cm.
2. Find the volume and total surface area of a cone with altitude 8 cm and area of the base 36π cm^2.
3. The lateral area of a right circular cone is 65π cm^2. Find the volume of the cone if its base area is 25π cm^2.
4. A solid right circular cone of height 3 cm and radius of the base 4 cm has the same volume as a right circular cylinder of height 1cm. Find
   (a) the ratio of the areas of their bases.
   (b) the ratio of their total surface areas.
5. The slant height of a right circular cone is 13 and the base radius is 12, then find the volume and the lateral area of the cone.
6. The volume of a right circular cone is $256\pi\ u^3$. If the radius of the base is $3u$, then find the altitude of the cone.
7. Find the lateral area of a right circular cone whose volume is $20\pi\ u^3$ and whose radius is $2u$.

KEY TERMS

- Apex
- Base
- Cone
- Cube and cuboid
- Cylinder
- Edges
- Lateral faces
- Parallelepiped
- Prism
- Polyhedrons
- Pyramid
- Regular Polyhedron
- Solid figure
- Surface area
- Tetrahedron
- Vertex
- Volume

Summary

1. Solid: A solid object is a three dimensional shape formed by a combination of planes and curved surfaces.

2. **Prism:** A prism is a three dimensional geometric figure in which the two ends called bases are congruent polygons on two parallel planes.

   The lateral faces of a prims are parallelograms.

3. **Types of prims**

   (i) Prisms can be classified as right prism or oblique prism.

   (ii) In aright prism, the lateral faces are rectangles.

   (iii) Prisms can be classified according to their bases as triangular prism, rectangular prism, …

4. **Lateral Area Formula**

   The lateral surface area of a right prism with perimeter p and height h is ph.

5. **Surface area formula**

   The surface area of a right prism with lateral area ph and base area A_B is $ph + 2A_B$.

6. **Cylinder**: A cylinder is a three dimensional figure formed by a combination of two equal circles called bases that are on two parallel planes and all line segments with end points on the bases that are parallel to the line segment joining the centers of their bases.

   Cylinder can be classified as right cylinder and oblique cylinder.

7. **Surface Area Formula**

   The surface area of right circular cylinder of height h and radius of the base r is $2\pi r^2 + 2\pi rh$.

8. **Volume Formula**

   The volume of a right circular cylinder of height h and radius of the base r is $\pi r^2 h$.

9. **Cone**

   A circular cone is a three dimensional geometric figure composed of a circular region called base and a point not on the plane of the base called apex or vertex together with all line segments joining every point of the base to the apex.

   Circular cones can be classified as right circular cone and oblique circular cone.

10. **Surface Area of Formula**

    The surface area of a right circular cone with radius of the base *r*, height *h* and slant height ℓ is $\pi r\ell + \pi r^2$.

11. **Volume Formula**

    The volume of a right circular cone with height *h* and radius of the base *r* is $\frac{1}{3}\pi r^2 h$.

12. **Pyramid**

    A pyramid is a polyhedron formed by all line segments whose end points are on a polygonal region called base to a point called apex or vertex which is not on the plane of the base.

13. **Surface area formula**

    The surface area of a pyramid is the sum of the area of the base and the area of the lateral faces i.e. $A_B + A_L$

14. **Volume Formula**

    The volume of a pyramid with base area A_B and height h is $\frac{1}{3}A_B h$.

Exercises

1. Find the volume and surface area of the following solids.

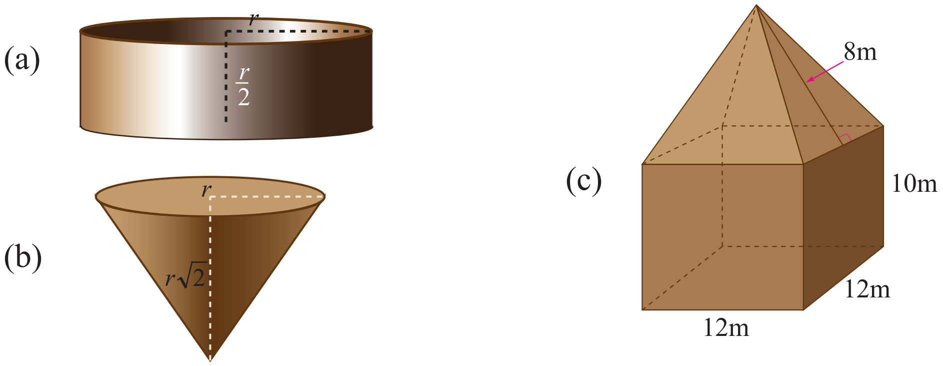


Figure 28.

2. Find the volume and surface area of a square pyramid whose edge of the base measuring 10 cm and whose height measuring 6 cm.
3. What is the volume of a regular square pyramid whose lateral faces are equilateral triangles of side 9 cm?

4. A regular square pyramid of height r is removed from a cone of radius r and height r. Find the volume of the remaining solid.

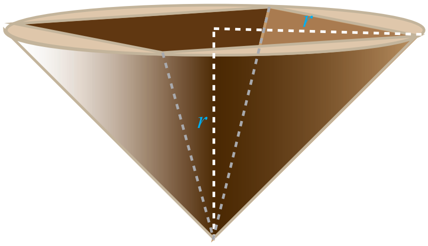


Figure 29.

5. The edges of the bases of a triangular pyramid of height 5 m are 5 m, 12 m and 13 m. Find the volume of the pyramid.
6. The diameter of the base of a right circular cone is 24 cm. If the slant height of the cone is 37 cm, find the volume and surface area of the cone.
7. The length of the edge of a 12.6 m tall regular hexagonal pyramid is 3.2 m. Find the volume and surface area of the pyramid.
8. A solid is made in the form of a cylinder surmounted by a right circular cone, as shown in the figure. The diameter of the base of the cylinder is 6 cm and its height is 6 cm. If the height of the cone is 4 cm, find the volume and surface area of the solid.

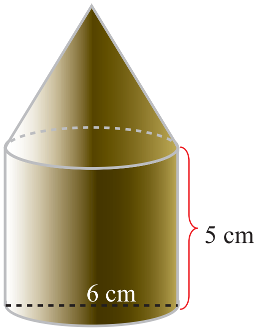


Figure 30.

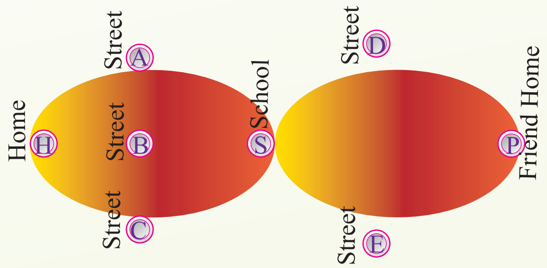
9. Find the volume of a regular square pyramid whose lateral faces are equilateral triangle of side $\sqrt{2}$ cm.
10. If a right pyramid has a square base with side 10 cm and a vertical height of 5 $\sqrt{3}$ cm, find its total surface area.

M12CH20

CHAPTER 20

PROBABILITY AND STATISTICS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

- fundamental counting principle;
- compute factorials;
- permutations;
- combinations;
- difference between permutations and combinations;
- sample space and events of an experiment;
- probability of an event, complementary events, mutually exclusive events, independent event, and conditional events, by Venn, tree diagrams, and contingency tables;
- odds of an event;
- calculate the expected value.

Introduction

Permutation and combinations are branches of mathematics that deal with arrangement and selection.

In this unit we will see the main differences between these two counting techniques and finally we will see the applications of counting techniques on calcualting probabilities. Different rules of finding probabilies will also be discussed in this unit.

Introduction to factorial notation

ACTIVITY 1

1. Find the product of the first
   (a) 4 natural numbers.
   (b) 5 natural numbers.
2. The product of the first 7 natural numbers is 5040, what is the product of the first 8 natural numbers? 6 natural numbers?
3. If the product of n natural numbers is k, what is the product of
   (a) $n + 1$ natural numbers?
   (b) $n - 1$ natural numbers? where $n > 1$.

DEFINITION

- Let $n \in \mathbb{W}$, the set of whole numbers the symbol "$n!$" is read as "n factorial".
- Let n be a natural number, then $n! = 1 \times 2 \times 3 \times \ldots \times n$ (the product of the first n-natural numbers)

EXAMPLE 1

(a) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$

(b) $4! = 1 \times 2 \times 3 \times 4 = 24.$

(c) $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720.$

Note

$0! = 1$

If n is a positive integer, then $n! = n(n - 1)!$

Permutations

Activity 2

How many 3 digit whole numbers can you form from the digits 6, 8, and 9 if a digit is used only once in a numeral?

Definition

- A permutation is an ordered arrangement of sequence of all or part of a finite set of things or objects. In permutation, order of objects is very important.
- If the objects are arranged in a line, then the permutation is said to be linear permutation or simply permutation,
- If the objects are arranged in the form of a circle, then the permutation is said to be circular permutation.

Example 2

How many arrangements of the letters A, B and C are there?

Solution

Consider the following possible order of arrangements of A, B and C.

1.	A, B and C	3.	B, A and C	5.	C, A and B
2.	A, C and B	4.	B, C and A	6.	C, B and A

Therefore, there are six different arrangements of the letters A, B and C.

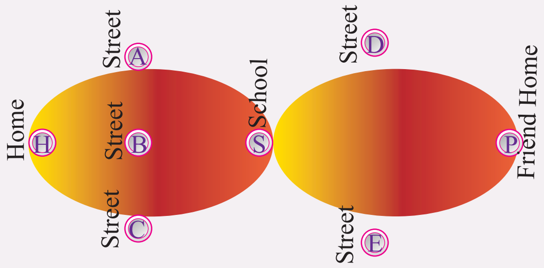
Example 3

Suppose there are three different ways from your home to your school and two different ways from your school to your friend's home. In how many different ways can you travel from your home to your friend's home through your school?

Solution

Let H, S and P be your home, school and friend's home respectively.

One possible way is to choose street A from H to S and then D from S to P. Another street is A from H to S and then E from S to P. Therefore we have six different permutations.



These are *A* and *D*, *A* and *E*, *B* and *D*, *B* and *E*, *C* and *D*, *C* and *E*.

Therefore, there are six different ways from your home to your friend's home through the school.

In general if there are M different ways from A to B and N different ways from B to C, then there are $M \times N$ **different ways** from A to C. This can be further extended as: If there are M_1 different ways from A_1 to A_2, M_2 from A_2 to A_3,...M_k from A_k to A_{k+1}, then there are $M_1 \times M_2 \times ... \times M_k$ different ways from A_1 to A_{k+1}.

EXAMPLE 4

How many two digit positive integers can be formed from the digits 4, 5, 6, 7, 8, 9, if:

(a) Each digit is used only once?

(b) There is no restrictions on the number of times each digit can be used?

Solution

(a) In this problem we will not have numbers like 55, 66,..etc. The first digit can be any of the digit 4, 5, 6, 7, 8, 9. Hence, there are six different ways of determining the first digit. If any one of these is taken to be the first digit, then the second digit can be taken only in five different ways.

Therefore, there are $6 \times 5 = 30$ two digit positive integers.

(b) In the above problem we didn't have the digits such as 44, 55, 66, 77, 88 and 99. Therefore, in this case we have all these integers together with the integers in question a.

Hence, the total of number of such digits is **30 + 6 = 36**

In fact in this problem each digit may be selected up to two times.

Therefore, there are six ways to write the first digit and also six ways to write the second. *A* total of

$6 \times 6 = 36$ different two-digit integers.

EXAMPLE 5

How many three digit odd natural numbers less than 600 can be formed from the integers 0, 1, 2, 3, 4, 5, 6, 7 ,8 and 9; if each integer is used only once?

Solution

The first digit cannot be zero because no three digit number starts by 0. Numbers like 015, 029, etc are not three digit integers. And the natural number to be formed is less than 600. Therefore, there are 5 possible ways to choose the first digit, 1, 2, 3, 4 or 5.

The second digit can be any one of these integers including 0 except the integer, which is taken to be the first digit. Therefore, there are 9 possible ways to choose the second digit. The third digit must be odd integer, since the number to be formed is odd. 1, 3, 5, 7, or 9.

The following table summarizes the above

Hundreds digit	Tens digit	Unit digit	Total three digit
Odd (3)	Odd (4)	Odd (3)	$3 \times 4 \times 3 = 36$
	Even (5)	Odd (4)	$3 \times 5 \times 4 = 60$
Even (2)	Odd (5)	Odd (4)	$2 \times 5 \times 4 = 40$
	Even (4)	Odd (5)	$2 \times 4 \times 5 = 40$

$\Rightarrow$ The required number of digits is 36 + 60 + 40 + 40 = 176.

- If there are n objects, then the number of arrangements that can be formed by taking r objects chosen from a set of n different objects is $\frac{n!}{(n-r)!}$; where $0< r \leq n$.

Verification (Illustration)

If there are r labels and n different objects, then we can have the following techniques of labeling the n objects in the r labels.

Example 6

The first label can be chosen in n different ways, the second in $n - 1$ and so on and finally the r label is $n-(r+1)$

1^{st}	2^{nd}	3^{rd}				r^{th}
n	$n - 1$	$n - 2$				$n - (r + 1)$

Therefore, we have a total of $n(n-1)(n-2) \ldots (n - (r + 1))$ arrangements.

$$\text{But } \frac{n!}{(n-r)!} = \frac{n(n-1)\ldots(n-r+1)(n-r)!}{(n-r)!}$$

$$= n(n-1)(n-2)\ldots(n-r+1).$$

Example 7

In how many ways can we order 11 books on a shelf that has 15 labels?

Solution

$$\frac{15!}{(15-11)!}=\frac{15!}{4!}=54486432000.$$

Example 8

A student buys seven different exercise books. In how many ways can he arrange four of them and use for each of his favorite four subjects?

Solution

Seven objects and 4 labels. $\frac{7!}{(7-4)!}=\frac{7!}{3!}=7\times6\times5\times4=480.$

Notation:

- Either of ${}_nP_r$, $P(n, r)$ can be used to denote the number of permutations if r objects are taken from n different objects.
- In this book we use the symbol ${}_nP_r$

1. Simplify the following:

(a) ${}_8P_6$	(c) ${}_nP_n$	(e) ${}_nP_r$
(b) ${}_{12}P_7$	(d) ${}_nP_{n-r}$	(f) ${}_nP_0$

Solution

$${}_8P_6=\frac{8!}{(8-6)!}=\frac{8\times7\times6\times5\times4\times3\times2\times1}{2\times1}=20160$$

$${}_{12}P_7=\frac{12!}{(12-7)!}=\frac{12!}{5!}=3991680$$

$${}_nP_n=\frac{n!}{(n-n)!}=\frac{n!}{0!}=n!$$

$${}_nP_r=\frac{n!}{(n-r)!}$$

$${}_nP_0=\frac{n!}{(n-0)!}=\frac{n!}{n!}=1!=1$$

$${}_nP_{n-r}=\frac{n!}{[n-(n-r)]!}=\frac{n!}{r!}$$

From this last two we see that $\frac{{}_nP_{n-r}}{(n-r)!}=\frac{{}_nP_r}{r!}$

2. Find the value of r or n in each of the following equations

(a) ${}_nP_2 = 12$ (b) ${}_5P_r = 20$ (c) ${}_nP_3 = 210$

Solution

(a) ${}_nP_2 = 12$

$$\Rightarrow \frac{n!}{(n-2)!} = 12 \Rightarrow n(n-1) = 12$$
$$\Rightarrow n^2 - n - 12 = 0$$
$$\Rightarrow n = 4, \text{ or } n = -3 \Rightarrow n = 4 \text{, since } n > 0$$

(b) ${}_5P_r = 20$ this means that

$r = 0, 1, 2, 3, 4, 5$

but ${}_5P_1, {}_5P_3, {}_5P_4, {}_5P_5 \neq 20$ and ${}_5P_2 = 20$

Therefore, $r = 2$

(c) ${}_nP_3 = 210$

$$\Rightarrow \frac{n!}{(n-3)!} = 210 \Rightarrow n(n-1)(n-2) = 210$$
$$\Rightarrow n^3 - 3n^2 + 2n = 210 \Rightarrow (n - 7)(n^2 + 4n + 30) = 0$$
$$\Rightarrow n = 7 \text{ and } n^2 + 4n + 30 > 0 \text{ for all } n \in \mathbb{N}$$
$$\Rightarrow n = 7.$$

Permutation of all elements; ${}_nP_n = n!$.

3. In how many ways can 10 different maths books be arranged on a shelf?

Solution

${}_{10}P_{10} = 10!$.

Permutations involving r elemetns, $0 < r < n$; ${}_nP_r$.

Activity 3

How many different words can we have from the letters of the word **T H A T?** For example HTAT is one of the words.

Now consider the number of possible arrangements of the letters of the word **FREEZER**

In this word there are three **E**'s and two **R**'s

The number of permutations of **F R_1 E_2 Z E_3 R_2** is **7!** But **E_1, E_2** and **E_3** can be arranged in **3!** ways and **R_1** and **R_2** can be arranged in **2!** Ways.

So the number of arrangements of **F$R_1$$E_1$$E_2$Z $E_3$$R_2$** can be found from the number of permutations of **FREEZER** which is $\frac{7!}{3!2!} = 420$

From this arrangements we can have the following generalizations.

1. The number of Permutations of N objects, P of which are identical is $\frac{N!}{P!Q!}$.

Example 9

The number of permutations of the letters DEGREE is

$\frac{6!}{3!}$ — 6! ← Number of letters; 3! ← Number of repeated letters

2. The number of permutations of N objects, P of which are identical and Q of which are identical (but different from the set of P objects) is $\frac{N!}{P!Q!}$

Example 10

The number of permutations of the letters RESPONSIBILITY is

$\frac{14!}{2!3!}$ — 14! ← Number of letters; 2!3! ← 2 S's and 3 I's

3. The number of permutations of N objects n_j of which are identical, where $j = 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, k$
$N = n_1 + n_2 + \ldots + n_k$ and the set of n_j objects is different from the set of n_i objects for all $j \neq i$ is $\frac{N!}{n_1!n_2!n_3!.........n_k!}$.

Example 11

On a shelf there are 30 books of which 10 are text books of mathematics for grade 10, 3 are text books of History for grade10, 7 are text books of Geography for grade 9, 5 text books of Physics for grade 9 and 5 different reference books of English. Then the total arrangement will be $\frac{30!}{10!3!5!7!1!1!1!1!1!}$.

Exercises

1. If four boys and two girls sit on a bench. In how many ways can they sit if the two girls
   (a) can sit next to each other?
   (b) do not sit next to each other?

2. Find the number of arrangements of the letters of the following words.

(a) EXIST	(d) EXPONENT
(b) EXAMPLE	(e) STATISTICS
(c) MATHEMATICS	(f) PARALLELOGRAM

Consider the set, $A = \{0, 1, 2\}$, all of the following sets are equal to set A.
$\{0, 1, 2\}, \{2, 0, 1\}, \{2, 1, 0\}, \{1, 0, 2\}, \{1, 2, 0\}$. This is because order of elements doesn't change a set.

Suppose you want to know the number of all subsets of a set of six elements. Some of the subsets will have three elements each. The number of ways of arranging the three objects from six objects is $6 \times 4 \times 5 = 120$.

The number of ways of arranging the three objects among themselves is 3! (Each to mean the same in the case of sets). Since the order of arrangement of elements of a set is not taken into consideration, the number of ways of calculating subsets of each with three elements from a s set of six elements is $= \frac{6 \times 4 \times 3}{3!} = 12$. A selection that can be formed by taking some or all of a finite set of thing (or objects) is called a combination.

Definition

If we have n different objects from which we select groups of r objects, the total number of possible groups (combinations) is given by:

$$\frac{\text{number of permutaons of } r \text{ objects from } n \text{ objects}}{\text{number of permutaons of } r \text{ objects among themselves.}}; 0 < r \leq n$$

$$= \frac{{}_nP_r}{r!}.$$

Suppose in a given school three students are running for three leadership positions. President, vice president and secretary. Each student has the same duty and grade. One can not be distinguished from the other in anyway. Here, the order in which the three leaders are assigned a position is not to be considered.

Hence, the number of different ways to select three leaders from 10 students is $\frac{{}_{10}P_3}{3!} = 120$.

Introduction to Combinations Notations

Let $n,\ r \in \mathbb{W}$ the set of whole numbers such that $n \geq r \geq 0$. Combinations are represented as ${}_nC_r$, $\binom{n}{r}$ or $C(n,r)$.

The symbol $C(n, r)$ or $\binom{n}{r}$ or ${}_nC_r$ is read as combination n, r or n-combination-r and each is equal to $\frac{n!}{r!(n-r)!}$.

EXAMPLE 12

Look at the simplified values of the following combiantions.

(a) $\binom{0}{0} = \frac{0!}{0!(0-0)!} = 1$

(b) $\binom{1}{1} = \frac{1!}{1!(1-0)!} = 1$

(c) $\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3!}{1!\ 2!} = 3$

(d) $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3!}{2!} = 3$

(e) $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = 35$

(f) $\binom{1}{0} = \frac{1!}{0!(1-0)!} = 1$

(g) $\binom{2}{0} = \frac{2!}{0!(2-0)!} = 1$

(h) $\binom{2}{1} = \frac{2!}{1!(2-1)!} = 2$

(i) $\binom{2}{2} = \frac{2!}{2!(2-2)!} = 1$

(j) $\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$

The difference between permutations and combiantions.

In permutation, the order is very important. However, in combination order doesn't matter. In combinations we are concerned only that the objects have been selected.

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ From this one can see that $\binom{n}{r} = \frac{{}_nP_r}{r\ !}$.

Note that $\binom{n}{r} \leq {}_nP_r$.

Combinations of r objects taken from n different objects at a time can also be written as $\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!}$.

For instance, $\binom{4}{2} = \frac{{}_4P_2}{2!} = \frac{4!}{2!(4-2)!} = 6.$

Example 13

1. In how many ways can a group of 5 students be chosen from a group of 12 students?

Solution

$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = 792$. This is because order doesn't matter in choice

2. In a class there are 12 students of which half are girls. Suppose a committee of 4 students is to be formed among them. Find the number of ways in which the committee must contain;

   (a) Exactly 2 boys (b) At least one boy (c) No boy

Solution

(a) If exactly 2 are boys, then exactly 2 will be girls. Therefore, 2 boys can be chosen from 6 boys in $\binom{6}{2}$ different ways and also 2 girls can be chosen from 6 girls in $\binom{6}{2}$ different way. Exactly 2 boys and exactly 2 girls can be chosen from 6 girls and 6 boys in $\binom{6}{2}\times\binom{6}{2} = 225$ different ways.

(b) At least one boy means greater than or equal to one boy.

Exactly one boys can be chosen in $\binom{6}{1}\times\binom{6}{3} = \frac{6\times6\times5\times4}{3!} = 120$ ways.

Exactly two boys can be chosen in $\binom{6}{2}\times\binom{6}{2} = 225$ ways.

Exactly three boys can be chosen in $\binom{6}{3}\times\binom{6}{1} = 120$.

Exactly four boys can be chosen in $\binom{6}{4}\times\binom{6}{0} = 15$ ways.

Therefore, at least one boy can be choose in a total of

120 + 225 + 120 + 15 = 480 ways. or $\binom{12}{4} - \binom{6}{4} = 495 - 15 = 480$.

(c) $\binom{6}{0}\binom{6}{4} = 15$ ways.

3. Find the total number of subsets of a set having 6 elements.

Solution

(i) the total number of subsets having 6 elements is $\begin{bmatrix}6\\6\end{bmatrix} = \frac{6!}{6!(6-6)!} = \frac{6!}{6!} = 1$.

(ii) Subsets with one element is $\binom{6}{1} = \frac{6!}{1!(6-1)!} = \frac{6!}{1!5!} = 6$.

(iii) Subsets with two elements $\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = 15$.

(iv) Subsets with three elements $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = 20$.

Because of symmetry $\binom{6}{4} = \binom{6}{2}, \binom{6}{5} = \binom{6}{1}, \binom{6}{6} = \binom{6}{0}$.

Therefore, the total number of subsets with elements four, five, and six is 15, 6, and 1 respectively.

Therefore, the total number of subsets of the set of six elements is

$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$.

Note

If a set has n-elements, then the number of subsets consisting of r elements is $\binom{n}{r}$. The total number of subsets of the set is $\binom{n}{0} + \binom{n}{n} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$.

EXAMPLE 14

In a group of k students, 25 students are chosen in 66 different ways. How many students are there in the group?

Solution

$\binom{k}{2} = 66$

$\Rightarrow \frac{k\ (k-1)\,(k-2)!}{2!(k-2)!} = 66$

$\Rightarrow k(k-1) = 132$

$\Rightarrow k^2 - k - 132 = 0$

$\Rightarrow (k+11)\,(k-12) = 0$

$\Rightarrow k = -11$ or $k = 12$.

Therefore, there are 12 students in the group.

Exercises

1. Simplify.

   (a) ${}_6P_2$ (d) ${}_nP_1$ (g) $\frac{{}_nP_k}{k!}; 0 < k < n$

   (b) ${}_5P_3$ (e) ${}_nP_4$

   (c) ${}_{11}P_4$ (f) $\frac{{}_{12}P_9}{9!}$

2. By using the digits 3, 4, 5, 6, 7, 8

   (a) How many three digit numerals can we form if we use a digit at most once in a numeral?

   (b) If we are allowed to repeat a digit how many three digit numerals can we form?

3. In how many ways can the letters of the word GARAGE be rearranged?

   (a) With out any restriction?

   (b) When the vowels are always together?

4. How many permutations of the word important end in *t*?

5. From seven books on a shelf if three are of one kind and the rest are of the second type. In how many different ways can we arrange the books on a shelf?

6. Find the total number of arrangements when 10 students sit around a circulate table.

7. If 5 students take a picture 3 students at a time sitting around a circular table, what is the total number of arrangements?

8. There are 11 girls in a committee. In how many different way can we assign a chairman and a secretary?

9. Compute

   (a) $C(12, 2)$ (c) $C(9, 1)$ (e) $\binom{n}{n-r} - \binom{n}{r}$

   (b) $C(9, 6)$ (d) $\binom{n}{1}$

10. What is the total number of arrangements for 12 different balls if there are only 8 places available?

11. If there are 3 green, 4 blue and 5 red balls where all the balls are the same except their colour, then what is the total number of permutations?

12. If $0 \le k \le n$, prove that

   (a) $\binom{n}{k} = \binom{n}{n-k}$ (b) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

13. If ${}_4P_n = 12$, what is the value of n?

14. If $\binom{n}{1} + \binom{n}{2} = 21$, find n;

15. If $\binom{n}{8} = \binom{n}{2}$, find n.

Introduction

In mathematics, calculating the uncertainty is said to be probability. Probability is a number expressing the chances for a given event will occur.

An event is defined as a basic term in probability.

Experiments, sample space and event

You might have tossed a fair coin or rolled a fair die or drawn a ball from a box that contains balls, all these processes are referred as experiments. An experiment is simply regarded as any process which has a well-defined results (or outcomes).

For example, in tossing a fair coin there are two possible outcomes, namely the occurrence of head or the occurrence of tail. In rolling a fair die, the possible outcomes are the occurrence of 1, 2, 3, 4, 5 or 6.

Trial: If an experiment is performed more than once, each repetition of the experiment that is performed under similar conditions is said to be trial.

For example, if you toss a coin 12 times, then each toss of the coin is a trial. Here, there are 12 trials. Also, if you roll a die 100 times, each roll is a trial.

Sample Space: The set of all possible outcomes of an experiment is said to be a sample space or possibility set.

Sample Point: An element of a sample space is said to be sample point.

Event: An event is any subset of a sample space.

Example 15

Suppose a coin is tossed, then the sample space is, $S = \{H, T\}$; where H is occurence of head and T is occurence of tail. H and T are sample points. The subsets of S are events.

Therefore, ϕ, $\{H\}$, $\{T\}$, $\{H, T\}$ are events.

Note

Here after by a coin or die we mean a fair coin or a fair die unless otherwise stated.

EXAMPLE 16

When you roll a die the sample space will be $\{1, 2, 3, 4, 5, 6\}$. Each of the numbers: 1, 2, 3, 4, 5 or 6 is a sample point.

Subsets of S such as: $\{1\}$, $\{2\}$, $\{1, 5\}$, $\{4, 5, 6\}$, etc. are events.

Remember that a set of n sample points has 2^n events.

Probability of an event

Let S be a sample space and $E \subseteq S$ be an event, The probability of E, denoted by $P(E)$ is defined as:

$$P(E) = \frac{n(E)}{n(S)}.$$

This definition can not be applied:

(i) if the outcomes are not equally likely

(ii) if the sample space is infinite

(iii) if the sample space is uncountable.

EXAMPLE 17

In a box there are 3 white, 4 red and 5 blue balls. All the balls are the same except for a colour. If one ball is drawn at random, find the probability that the ball chosen will be

(a) white
(b) red
(c) blue
(d) yellow
(e) not red
(f) not green

Solution

Let W, R, B, Y and G be the set of white, red, blue, yellow and green balls respectively. Then $n(W) = 3$, $n(R) = 4$, $n(B) = 5$, $n(Y) = 0$, and $n(G) = 0$.

The sample space S has $3 + 4 + 5 = 12$, sample points. Hence,

(a) $P(W) = \frac{3}{12} = \frac{1}{4}$

(b) $P(R) = \frac{4}{12} = \frac{1}{3}$

(c) $P(B) = \frac{5}{12}$

(d) $P(Y)=0$, since there are no yellow balls in the box.

Here, Y is said be null event.

(e) $P(B \cup R) = \frac{5+4}{12} = \frac{3}{4}$.

(f) P (not green)$= \frac{12}{12} = 1.$ because all balls are not green. This is said to be sure event or absolute event.

Note

$0 \le P(E) \le 1$, $P(\phi) = 0$ and $P(S) = 1$.

Types of events

In this section we are going to see the following types of events:

(i) Simple and compound events

(ii) Complement of an event

(iii) Mutually exclusive events

(iv) Exhaustive event

(v) Dependent and independent events.

Simple and compound events

A. Simple event (or ordinary event)

An event E that has exactly one element is said to be a simple event.

B. Compound Event

An event E that has two or more elements is said to be a compound event.

Example 18

If you toss a coin, then $S = \{H, T\}$

$\{H\}$ and $\{T\}$ are simple events.

$\{H, T\}$ is a compound event.

Example 19

If you roll a die, then the following are simple events. $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$. The following are some of the compound events $\{1, 2\}, \{1, 2, 3\}, \{1, 4, 5, 6\}, \{2, 3, 4, 5, 6\}$.

Complementary events

Two events are said to be complementary events when one occurs if and only if the other does not. The probabilities of two complementary events add up to 1.

An event and its complement are complementary events.

Complement of an event

Let E be an event, the complement of E, denoted by E' is the event consisting of all sample space outcomes that do not correspond to the occurrence of E. $P(E')$ is the probability that E will not occur.

The rule of complements

Note that $E \cup E' = S$ and $E \cap E' = \phi$.

Therefore, $P(E) + P(E') = 1$.

$\Rightarrow P(E') = 1 - P(E)$.

EXAMPLE 20

1. If a die is rolled, find the probability that the die does not show an odd number.

Solution

$S = \{1, 2, 3, 4, 5, 6\}$.

Let $E = \{1, 3, 5\}$, the set of odd numbers on the die, then

$$P(E) = \frac{3}{6} = \frac{1}{2} \Rightarrow P(E') = 1 - \frac{1}{2} = \frac{1}{2}.$$

2. A pair of dice is tossed once, find the probability that the sum of the numbers showing up is at least 5.

Solution

Look at the following addition table.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$S = \{(1, 1), (1, 2), \ldots, (6,6)\}$.

Let $E = \{(x, y): x + y < 5; x, y \in \{1, 2, 3, 4, 5, 6\}\}$

Then, $E' = \{(x, y): x + y \geq 5; x, y \in \{1, 2, 3, 4, 5, 6\}\}$.

From the table one can see that $n(E) = 6$ and $n(S) = 36$.

Hence, $P(E') = 1 - \frac{6}{36} = \frac{5}{6}$.

The addition rule

Let A and B be events, the probability that A or B (or both) will occur denoted by $P(A\cup B)$ is given by:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B).$$

EXAMPLE 21

Suppose a die is rolled, what is the probability that it will show either a prime number or an even number?

Solution

$S=\{1, 2, 3, 4, 5, 6\}$.

Let $E_1 = \{2, 3, 5\}$ Prime, $E_2 = \{2, 4, 6\}$ Even

$E_1\cap E_2 = \{2\}$

$P(E_1\cup E_2) = P(E_1) + P(E_2) - P(E_1\cap E_2)$

$$= \frac{3}{6}+\frac{3}{6}-\frac{1}{6}=\frac{5}{6}.$$

Mutually exclusive events

Two events A and B are mutually exclusive events if they have no sample space outcomes in common; A and B cannot occur simultaneously so that $P(A\cap B)= 0$.

In general, events $E_1, E_2, E_3, \ldots, E_n$ are said to be mutually exclusive if the occurrence of any one of them automatically implies that the remaining $(n-1)$ events will not occur.

ACTIVITY 4

In rolling a die, which of the following events can occur together?

(a) An even number and an odd number.

(b) An even number and multiple of 3.

(c) An odd number and multiple of 5.

Note:

$$E_i\cap E_j=\phi \text{ for all } i\neq j,\ i, j = 1, 2, 3, \ldots, n \text{ so that } P(E_i\cap E_j) = 0 \text{ for all } i\neq j.$$

EXAMPLE 22

Consider the set of students in your class.

Let E_1 = the set of female students

E_2 = the set of male students

E_1 and E_2 are mutually exclusive events. Therefore, $E_1\cap E_2=\phi$.

Example 23

Suppose two cards are drawn from a pack of 52 playing cards. Consider the following events.

B = the set of black cards K = the set of kings

R = the set of red cards J = the set of jacks.

Q = the set of queens

Find some mutually exclusive events.

Solution

(a) B and R are mutually exclusive because there is no card which is both black and red.

(b) Q, K and J are mutually exclusive.

(c) B and Q are not mutually exclusive because there is a queen on a black card.

The addition rule for mutually exclusive events

Remember that two events E_1 and E_2 are said to be mutually exclusive if the occurrence of E_1 excludes the probability of the occurrence of E_2. i.e either E_1 or E_2 but not both can occur.

Example 24

Suppose a number is chosen at random from the set

$S = \{1, 2, 3, 4, \ldots, 9\}$.

Let A, B, C and D be sets containing the chosen numbers that are prime, odd, even and multiples of 5 respectively. Then,

$A = \{2, 3, 5, 7\}$ $C = \{2, 4, 6, 8\}$

$B = \{1, 3, 5, 7, 9\}$ $D = \{5\}$

Observe that B and C are mutually exclusive as well as C and D are mutually exclusive because $B \cap C = \phi$ and $C \cap D = \phi$.

But, A and B, A and D, B and D, and also A and C are not mutually exclusive. Because $A \cap B = \{2, 3, 5, 7\}$, $A \cap D = \{5\}$, $B \cap D = \{5\}$ and $A \cap C = \{2\}$.

Exhaustive Events

Events are said to be exhaustive events if their union is equal to the sample space. i.e Event $E_1, E_2, E_3, \ldots, E_n$ are exhaustive events if $E_1 \cup E_2 \cup \ldots \cup E_n = S$.

Example 25

In tossing a coin $\{H\}$ and $\{T\}$ are exhaustive events because their union gives the sample space.

Example 26

Suppose a die is rolled. Then

(a) {1}, {2}, {3}, {4}, {5}, {6} are exhaustive events because their union is {1, 2, 3, 4, 5, 6} which is the sample space.

(b) {1}, {2, 3, 4}, {1, 5, 6} are also exhaustive events.

(c) {2, 3, 4}, {5, 6}, {4}, {3} are not exhaustive events because their union doesn't include 1.

- If $E_1, E_2, E_3, \ldots, E_n$ are events such that $E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$ and $E_i \cap E_j = \phi$ for all $i \neq j$; $j = 1, 2, 3, \ldots, n$, then the events are said to be exhaustive and mutually exclusive events.

Example 27

(a) {1}, {2}, {3}, {4}, {5}, {6} are exhaustive and mutually exclusive events. Because their union gives the sample space and the intersection of any distinct events is empty set.

(b) {1, 2}, {3, 4}, {4, 5, 6} are exhaustive but not mutually exclusive because $\{3,4\} \cap \{4,5,6\} = \{4\} \neq \varnothing$.

(c) {1, 2}, {5, 6} are mutually exclusive but not exhaustive because $\{1, 2\} \cup \{5, 6\} \neq S$.

Independent and dependent events

A. Independent Events:

Two events are independent if the occurrence or non-occurrence of one event does not affect the occurrence or non-occurrence of the other event. i.e Events *A* and *B* are independent if neither the occurrence of *A* nor the occurrence of *B* has any effect on the other.

Example 28

If a coin is tossed and a die is rolled, then the occurrence or non-occurrence of a head on the coin doesn't affect the occurrence or non-occurrence of a number (say 2) on the die.

B. Dependent Events:

Two events are dependent if the occurrence or non – occurrence of one affects the other.

EXAMPLE 29

Suppose there are 5 blue and 3 red balls in a bag. If a blue ball is taken out and not replaced back (Without are placement), then the remaining balls are 7 where 4 are blue and still 3 are red. The probability of the first ball to be blue is $\frac{5}{8}$. Whereas the probability of the second ball to be blue is $\frac{4}{7}$ or the probability of the second ball to be red is $\frac{3}{7}$ instead of $\frac{3}{8}$ like at the beginning.

Therefore, the selection of the second ball depends on the first selection.

Note

If the ball is replaced (with replacement) and then a second ball is taken out then the two events are independent.

Multiplication Rule

If A and B are independent events, then $P(A \cap B) = P(A)\,P(B)$.

EXAMPLE 30

(a) If a coin is tossed and a die is rolled, find the probability that

(i) The coin shows head and the die shows number 5.

(ii) The coin shows tail and the die shows a number less than 4.

Solution

(i) $P(H) = \frac{1}{2}$ and $P(\text{number } 5) = \frac{1}{6}$

$$P(H \text{ and } 5) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}.$$

(ii) Let $E = \{1, 2, 3\}$; a number on the die less than 4

$$P(T) = \frac{1}{2},\ P(E) = \frac{3}{6}$$

$$P(T \text{ and } E) = \frac{1}{2} \times \frac{3}{6} = \frac{1}{4}.$$

(b) If A and B are independent events with $P(A) = \frac{3}{5}$ and $P(B) = \frac{5}{7}$, find

(i) $P(A \cap B)$ (ii) $P(A' \cap B)$

Solution

(i) $P(A \cap B) = P(A)P(B) = \frac{3}{5} \times \frac{5}{7} = \frac{3}{7}.$

(ii) $P(A') = 1 - P(A) = 1 - \frac{3}{5} = \frac{2}{5}$

$$\Rightarrow P(A' \cap B) = P(A')P(B) = \frac{2}{5} \times \frac{5}{7} = \frac{2}{7}.$$

Multiplication rule for k independent events

If $E_1, E_2, E_3, \ldots, E_k$ are independent events, then
$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_k) = P(E_1)P(E_2)P(E_3)\ldots P(E_k)$.

EXAMPLE 31

1. In a box there are 3 white, 7 red and 8 blue balls. All the same except colour.
   (a) Three balls are drawn at random one after the other with replacement. Find the probability that
      (i) The first is white, the second is red and the third is blue.
      (ii) All are red.
      (iii) The first two are blue and the third is white
   (b) Three balls are drawn at random one after the other without replacement. Find the probability that.
      (i) The first is white, the second is red and the third is blue.
      (ii) All are red
      (iii) The first two are blue and the third is white
      (iv) The first is blue, the second is red and the third is white.

Solution

(a) There are 3 + 7 + 8 = 18 balls.

$P(\text{white}) = \frac{3}{18}$, $P(\text{red}) = \frac{7}{18}$ and $P(\text{blue}) = \frac{8}{18}$.

Since the balls are replaced after each draw,

(i) $P = \frac{3}{18} \times \frac{7}{18} \times \frac{8}{18} = \frac{7}{243}$

(iii) $\frac{8}{18} \times \frac{8}{18} \times \frac{3}{18} = \frac{8}{243}$.

(ii) $P = \frac{7}{18} \times \frac{7}{18} \times \frac{7}{18} = \frac{343}{5832}$

(b) If the first ball drawn is white and if it is not replaced, then the number of white balls is decreased by 1 and the total number of balls is decreased by 1 and will be 17. If the second ball drawn is red and if it is not replaced the number of red balls

is decreased by 1 and the total number of balls will be 17 – 1 = 16 and so on.
Hence,

(i) P (white followed by red followed by blue) $= \frac{3}{18} \times \frac{7}{17} \times \frac{8}{16} = \frac{7}{204}$

(ii) P (the first ball is red) $= \frac{7}{18}$

P (the second ball is red) $= \frac{6}{17}$

P (the third ball is red) $= \frac{5}{16}$

Hence P (all red) $= \frac{7}{18} \times \frac{6}{17} \times \frac{5}{16} = \frac{35}{816}$.

(iii) P(blue followed by blue followed by white) $= \frac{8}{18} \times \frac{7}{17} \times \frac{3}{16} = \frac{7}{204}$.

(iv) P(blue followed by red followed by white) $= \frac{8}{18} \times \frac{7}{17} \times \frac{3}{16} = \frac{7}{204}$.

Addition rule for k mutually exclusive events

The events $A_1, A_2, A_3, \ldots, A_k$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_k).$$

EXAMPLE 32

In a playing card. Let J, K and Q be the sets consisting of jack, king and queen cards. Find $P\,(J \cup K \cup Q)$.

Solution

J, K and Q are mutually exclusive cards.

Thus, $P\,(J \cup K \cup Q) = P(J) + P(K) + P(Q)$

$$= \frac{4}{52} + \frac{4}{52} + \frac{4}{52}$$

$$= \frac{3}{13}.$$

Conditional probability

The probability of an event A given the condition that the event B has occurred written as $P(A \mid B)$ and read as the probability of A given B is said to be the conditional probability of A given B.

The formula for conditional probability is:

$$P(A|B)=\frac{P(A\cap B)}{P(B)};\ \text{where } P(B) > 0.$$

Also, the conditional probability that B will occur given that A will, is given by

$$P(B|A)=\frac{P(A\cap B)}{P(A)};\ \text{where } P(A) > 0.$$

Note

(i) $P(A\cap B) = P(A|B)\,P(B)$ and also, $P(A\cap B) = P(B|A)\,P(A)$

$\Rightarrow P(A|B)\,P(B) = P(B|A)\,P(A)$.

(ii) $P(A|B) \neq P(B|A)$.

(iii) If A and B are independent events, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

(iv) If A and B are mutually exclusive events, then $P(A|B) = 0$.

Example 33

A student took two exams, Mathematics and English. If the probability that the student will pass English exam is 0.8 and the probability that the student will pass both Mathematics and English exams is 0.6, find the probability that the student will pass Mathematics exam given that the student passed English exam.

Solution

Let $P(M)$ be the probability that the student will pass Mathematics exam.

$P(E)$ the probability that the student will pass English exam.

Then $P(M|E)=\dfrac{P(M\cap E)}{P(E)}=\dfrac{0.6}{0.8}=\dfrac{3}{4}=0.75$.

Example 34

A card is selected at random from a pack of 52 playing cards. What is the probability that it is a jack of heart given that it is a red card?

Solution

$$P\text{ (jack of heart/red)} = \frac{P(\text{jack of heart and red})}{P(\text{red})}=\frac{\left(\frac{1}{52}\right)}{\frac{1}{2}}=\frac{1}{26}.$$

Remember that, $P(\text{red})=\dfrac{1}{2}$ and there is only one jack of heart.

Example 35

Suppose a fair die is tossed once, what is the probability of

(a) an even number given that it is prime?

(b) an odd number given that it is composite?

(c) a prime number given that it is

(i) Even? (ii) Odd?

(d) getting an odd number first and then a prime number?

Solution

$S = \{1, 2, 3, 4, 5, 6\}$.

Let $E = \{2, 4, 6\}$, the set of even numbers in S.

$P = \{2, 3, 5\}$, the set of prime numbers in S

$O = \{\text{the number is odd}\} = \{1, 3, 5\}$, the set of odd numbers in S

$C = \{\text{the number is composite}\} = \{4, 6\}$

The only number that is even and prime is 2.

(a) $P(E|P) = \dfrac{P(E \cap P)}{P(P)} = \dfrac{\left(\frac{1}{6}\right)}{\left(\frac{3}{6}\right)} = \dfrac{1}{3}.$

(b) There is no number in S which is odd and composite

$$P(O/C) = \frac{0}{\left(\frac{3}{6}\right)} = 0.$$

(c)

(i) $P(P|E) = \dfrac{P(P \cap E)}{P(E)} = \dfrac{\left(\frac{1}{6}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{3}.$ (ii) $P(P|O) = \dfrac{P(P \cap O)}{P(O)} = \dfrac{\frac{2}{6}}{\frac{1}{2}} = \dfrac{2}{3}.$

Example 36

There are 400 students in a school. 200 are enrolled in Mathematics club and 180 students are enrolled in Physics club. There are 60 students who are enrolled in both Mathematics and Physics clubs. Find the probability that

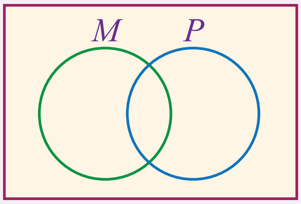
(a) the student is enrolled in Mathematics club given that he is enrolled in Physics club.

(b) the student is enrolled in Mathematics club given that he is en rolled in Mathematics club.

(c) the student is not enrolled in Mathematics club given that he is enrolled in Physics club.

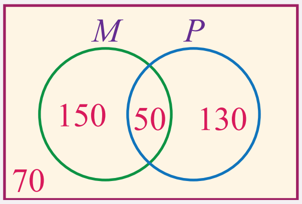
(d) the student is not enrolled in Physics club given that he is not enrolled Mathematics club.

Solution



Let us use a Venn diagram inorder to answer these questions. Let M represent the number of students enrolled in Mathematics club and P represents the number of students enrolled in Physics club.

From set operations we know that,



$n(M \cup P) = n(M) + n(P) - n(M \cap P)$

$n(M) = 200$, $n(P) = 180$, $n(M \cap P) = 50$

$n(M') = 400 - 200 = 200$

$n(P') = 400 - 180 = 220$

(a) $P(M|P) = \frac{50}{180} = \frac{5}{18}$

(b) $P(P|M) = \frac{50}{200} = \frac{1}{4}$

(c) $P(M'|P) = \frac{130}{180} = \frac{13}{18}$

(d) $P(P'|M') = \frac{70}{200} = \frac{7}{20}$

EXAMPLE 37

The following contingency table shows the relationship between weight and height of a group of individuals in a certain entertenment group.

Weight/Height	Tall	Medium	Short	Raw total
Obese	30	45	15	90
Normal	35	85	45	165
Under weight	20	40	15	75
Column total	85	170	75	330

Find the probability that a randomly chosen individual from the group is

(a) Tall given that the individual is obese.

(b) Normal given that the individual is Tall.

(c) Obese given that the individual is short.

(d) Short given that the individual is under weight.

(e) Tall given that the individual is under weight.

Solution

From the table one can see that

(a) $P(\text{Tall}\,|\,\text{obese}) = \dfrac{30}{90} = \dfrac{1}{3}$.

(b) $P(\text{Normal}\,|\,\text{Tall}) = \dfrac{35}{85} = \dfrac{7}{17}$.

(c) $P(\text{obuse}\,|\,\text{short}) = \dfrac{15}{75} = \dfrac{1}{5}$.

(d) $P(\text{short}\,|\,\text{under weight}) = \dfrac{15}{75} = \dfrac{1}{15}$.

(e) $P(\text{Tall}\,|\,\text{under weight}) = \dfrac{20}{75} = \dfrac{4}{15}$.

Odds in favor and odds against

Odds: The ratio that compares the number of favorable outcomes of an event to the number of unfavorable outcomes is said to be odds.

Odds in favor: odds in favor of an event$=\dfrac{\text{number of favorable outcomes}}{\text{number of unfavorable outcomes}}$.

Odds against: odds against an event $=\dfrac{\text{number of unfavorable outcomes}}{\text{number of favorable outcomes}}$.

EXAMPLE 38

If a fair die is tossed, find

(a) The odds in favor of turning up a 6.

(b) The odds against turning up a 6.

Solution

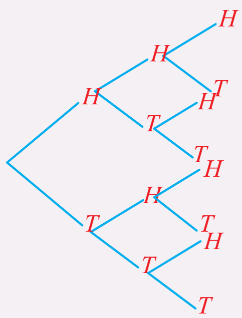
$S = \{1, 2, 3, 4, 5, 6\}$, $E = \{6\}$, $E' = \{1, 2, 3, 4, 5\}$

(a) Odds in favor of $E = \dfrac{n(E)}{n(E')} = \dfrac{1}{5}$.

(b) Odds against $E = \dfrac{n(E')}{n(E)} = \dfrac{5}{1}$.

Example 39

What is the odds in favor of getting exactly 2 heads when three coins are tossed.



Solution

Look at the following tree diagram

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$E = \{HHT, HTH, THH\}$

$\Rightarrow$ odds in favor of $E = \dfrac{n(E)}{n(E')} = \dfrac{3}{5}$.

Example 40

The odds against an event are 3:5, what is the probability that the event will occur?

Solution

Let E be the event, then $\dfrac{n(E')}{n(E)} = \dfrac{3}{5} \Rightarrow n(E') = \dfrac{3}{5}n(E)$.

$$P(E) = \frac{n(E)}{n(E \cup E')} = \frac{n(E)}{n(E)+n(E')} = \frac{n(E)}{n(E)+\frac{3}{5}n(E)}$$

$$= \frac{5n(E)}{8n(E)} = \frac{5}{8}.$$

Note: The difference between odds and probability are odds of an event are ratio of success to failure where as probablity of an event is the ratio of the success to the sum of success and failure odds in favour of event $A = \dfrac{P(A)}{1-P(A)}$.

odd agaisnt events $= \dfrac{1-P(A)}{P(A)}$.

Exercises

1. A coin is tossed and a die is rolled. Find the probability that
   (a) The coin shows head and the die shows 6.
   (b) The coin shows tail and the die shows an even number.
2. A card is chooses at radom from a deck of 52 playing cards and a second card is chosen. What is the probability of choosing a jack and then number 7
   (a) With replacement. (b) without replacement

3. A jar contains 3 red, 5 green, 4 blue and 6 white balls. If 3 balls are chosen at random one after the other, determine the probability of each of the following events
   (i) with replacement (ii) without replacment
      (a) The fist is red, the second is green and the third is whte
      (b) The fist two are blue and the third is green
      (c) All are white.
4. It is found that 8 out of 10 students used pencils in maths class. If students are chosen one after the other with replacement, what is the probability that all the four students use pencils in maths class?
5. When two dice are rolled, find the probability of getting
   (a) Sum 3 (b) Sum greater than 5.
   (c) Product 12
   (d) Product between 6 and 18 exclusive.
6. If a fair die is rolled 3 times, find the probability that
   (a) Each time the die shows 5
   (b) Exactly two times the die shows 3
   (c) Exactly two times the die shows 2 and one time 4.
7. Which of the following can not be the probability of an event.
   (a) 0 (b) 2 (c) $\frac{1}{\sqrt{2}}$ (d) e
8. A bag contains 4 red and 6 black balls. If two balls are drawn at random, find the probability that
   (a) Both balls are red (c) At least one ball is black
   (b) Exactly one ball is black (d) At most one ball is red
9. A bag contains 4 red and 6 black balls. If two balls are drawn at random one after the other
   (i) with replacement
   (ii) without replacement, find the probability that
      (a) The first is red and the second is red.
      (b) The first is black and the second is red
      (c) The first is black no mather about the colour of the second
      (d) The first can be any colour but the second must be black.

10. Three men M_1, M_2 and M_3 are firing at a target independently and have probabilities 0.2, 0.25 and 0.3 respectively. What is the probabilitiy that
    (a) exactly one of them hit the target?
    (b) exactly two of them hit the target?
    (c) exactly three of them hit the target?
    (d) at least one of them hit the target?
    (e) at most one of them hit the target?
    (f) at most two of them hit the target?
    (g) at least two of them hit the target?

Using combinations to solve probability

Example 41

In a box of similar red and blue pens there are 3 red and 2 blue pens. If two pens are selected at random, find the probability that

(a) all are red pens

(b) one red and one blue pen.

(c) both pens have the same colour

Solution

(a) The probability that all are red pens is

- Selection of 2 out of 5 disregarding order is $C(5, 2)$
- Selection of 2 reds from 3 reds is $C(3, 2)$ (*Here no blue will be selected*)

$$\frac{C(3,2)\times C(2,0)}{C(5,2)}=\frac{3\times 1}{10}=\frac{3}{10}.$$

(b) The probability to select one red and one blue pen is

$$\frac{C(3,1)\times(2,1)}{C(5,2)}=\frac{3\times 2}{10}=\frac{3}{5}.$$

(c) The probability to choose balls of the same colour (two red or two blue)

$$\frac{C(3,2)}{C(5,2)}+\frac{C(2,2)}{C(5,2)}=\frac{3}{10}+\frac{1}{10}=\frac{4}{10}=\frac{2}{5}.$$

Example 42

In a box there are 3 red balls, 5 yellow ball and 6 green balls. All balls are identical except their colour. If 3 balls are drawn at random, find the probability that

(i) All are red balls

(ii) Exactly 2 are yellow balls.

(iii) Exactly 2 are yellow balls and one ball is green.

(iv) At least one is green ball.

Solution

In question a the 3 balls are drawn at the same time. In questions *b* and *c* the balls are drawn one after the other.

(a) The total number of balls is 3 + 5 + 6 = 14. 3 - balls are drawn from 14 - balls in $\binom{14}{3}$ ways.

(i) 3 red balls are drawn from 3 red balls in $\binom{3}{3}$ ways. There is no ball drawn from the reaming 14 – 3 = 11 balls.

Hence the probability that all are red is $\dfrac{\binom{3}{3}}{\binom{14}{3}}$.

(ii) Two yellow balls are chosen from 5 yellow balls means the other 1 ball is chosen from the remaining 14 – 5 = 9 balls.

This can be done in $\binom{5}{2}\binom{9}{1} = \dfrac{5!}{2!3!} \times \dfrac{9!}{1!8!} = 10 \times 9 = 90$ ways.

Hence the probability that exactly 2 are yellow balls is $\dfrac{\binom{5}{2}\binom{9}{1}}{\binom{14}{3}} = \dfrac{45}{182}$.

(iii) By a similar reason, the required probability is $\dfrac{\binom{5}{2}\binom{6}{1}}{\binom{14}{3}} = \dfrac{15}{91}$.

(iv) At least one ball is green is the complement of there is no green ball. That is the 3 balls are chosen from 14 – 6 = 8 balls. This can be done in $\binom{8}{3}$ ways.

Hence the probability of drawing at least one green ball is found to be;

$$1\text{-}P(\text{no green ball}) = 1 - \frac{\binom{8}{3}}{\binom{14}{3}} = 1 - \frac{56}{364} = \frac{87}{91}.$$

EXERCISES

1. In a box there are 3 blue and 5 white marbles, if 2 marbles are selected at random what is the probability to see
   (a) marbles of the same colour?
   (b) marbles of different colours?
   (c) both blue marbles?
   (d) both white marbles?
2. If 5 friends meet on a road. How many handshakes do we expect?
3. If the probability for Abebe to go to school is ½ and the probability that he will wear his new cloth is 1/5, then is the probability that Abebe will go to school by wearing his new cloths?
4. A committee of 3 members is formed from 8 students in which 3 of them are girls. Find the probability that the committee members
   (a) all are boys
   (b) all are girls
   (c) 2 of them are boys
   (d) at least 2 are girls.

Weighted Average: Weighted average is average or mean of k - numbers $n_1, n_2, n_3, ..,$ n_k with weights $w_1, w_2, w_3, ..., w_k$ respectively given by $\dfrac{w_1n_1 + w_2n_2 + w_3n_3 + ...w_kn_k}{w_1 + w_2 + w_3 + ... + w_k}$

That is the average of the distribution given by the following table.

Number	weight
n_1	w_1
n_2	w_2
n_3	w_3
.	.
.	.
.	.
n_k	w_k

$$\text{The average weight} = \frac{\sum_{i=1}^{k} n_i w_i}{\sum_{i=1}^{k} w_i}.$$

Activity 5

1. Find the weighted average of the following distributions.

Number	weight
4	1
5	2
6	5
7	4
8	4
9	3
10	1

2. Fill in the following table

Numbers x_i	frequency (*f*) *fi*	Relative frequence $\left(\frac{f}{N}\right)$	$x_i\left(\frac{f}{N}\right)$
4	1	$\frac{1}{20}$	
5	2	$\frac{2}{20}$	
6	5		
7	4		
8	4		
9	3		
10	1		
	$N = 20$	Total =	Total =

Expected value

The expected value of a random variable is the weighted average of its probability distribution.

Example 43

Suppose that when you roll a die, you will win L$ 3 for a number greater than 4 and loss L$1 for a number less than 5. Find the expected value of money you win or loss for one roll of the die.

Solution

The sample space, $S = \{1, 2, 3, 4, 5, 6,\}$

Roll (x_i)	1	2	3	4	5	6
Probability ($P(x_i)$)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Amount (L$)	−1	−1	−1	−1	3	3

The weighted average is:

$$(-1)\left(\frac{1}{6}\right) + (-1)\frac{1}{6} + (-1)\frac{1}{6} + 3\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) = -\frac{4}{6} + \frac{6}{6} = \frac{2}{6} = \frac{1}{3}.$$

Thus, the expected value is $\frac{1}{3}$

That is the average money you get is L$ $\frac{1}{3}$ per roll.

The formula for expected value

The expected value, denoted by $E[X]$ of a discrete random variable X with finite number of possible outcome

$\{x_1, x_2, x_3, \ldots x_k\}$ and $P(x_i)$ the probability of x_i is given by the formula

$$\mathrm{E}[\mathrm{X}] = \sum_{i=1}^{k} x_i \mathrm{P}(x_i)$$

$$E[X] = x_1P(x_1) + x_2P(x_2) + \ldots + x_kP(x_k).$$

Example 44

A girl works at a book shop as sales person. If the probability that she does not sell a book is 0.01, and for the probability that she sells, 5, 10, 20, 50 and 65 books per day the expected value of the number of books she sells per day are 0.05, 0.1, 0.25, 0.44, and 0.15 respectively, find the expected daily sales.

Solution

The Number of books sold	0	5	10	20	50	65
Probability	0.01	0.05	0.1	0.25	0.44	0.15

$$\begin{aligned} E[X] &= 0(0.01) + 5(0.05) + 10(0.1) + 20(0.25) + 50(0.44) + 65(0.15) \\ &= 0 + 0.25 + 1 + 5 + 22 + 9.75 \\ &= 38. \end{aligned}$$

The girl, on the average, expects to sell 38 books per day.

Example 45

Find the expected number of boys for a family of three children.

Solution

Let b represents a boy and g represents a girl. Then the sample space is

$$S = \{ggg, ggb, gbg, bgg, bgb, gbb, bbg, bbb\}$$

The probability of No boy is equal to the probability of three girls which is $\frac{1}{8}$ as shown in the following table.

Number of boys x_i	Probability $p(x_i)$	$x_i\, p(x_i)$
0	$\frac{1}{8}$	$0\left(\frac{1}{8}\right)$
1	$\frac{3}{8}$	$1\left(\frac{3}{8}\right)$
2	$\frac{3}{8}$	$2\left(\frac{3}{8}\right)$
3	$\frac{1}{8}$	$3\left(\frac{1}{8}\right)$

$$E[x] = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1.5$$

The expected value of the number of boys is 1.5.

Example 46

Suppose there are two projects. Project A and project B such that project A shows a probability of 0.7 to achive a value of L\$ 5,000,000 and a probability of 0.3 to achive a value of L\$ 7,5000,000 up on completion.

Project B shows a probability of 0.4 to achive L\$ 6,000,000 and a probability of 0.6 to achive L\$ 5,800,000 upon completion.

Which project has more promising future development?

Solution

In order to select the promising project, one need to calcualte the expected value of each project.

$$E[A] = 0.7 \times 5{,}000{,}000 + 0.3 \times 7{,}500{,}000 = 5{,}750{,}000$$

$$E[B] = 0.4 \times 6{,}000{,}000 + 0.6 \times 5{,}800{,}000 = 5{,}880{,}000.$$

Clearly, $E[B] > E[A]$. Therefore, project B should be selected.

Exercises

1. A fair coin is tossed three times. Let x be the number of faces that show head. Find the expected value.
2. A fair die is tossed twice.

   Let X be the number of faces that show an even number. Copy and fill in the following table. Determine the sample space S and calculate the expected value.

   $S = \{(1, 1), (1, 2), ..., (6, 6)\}$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)			
2				(2, 4)		
3						
4						
5						
6						(6, 6)

3. A team of youth association in a village visit HIV - positive people one day, two days, three days or no days every week. The probability that the team visits them no day is 0.1, one day is 0.4, two days is 0.3, and three days is 0.2.

   Find the expected value.
4. A men's soccer team plays soccer zero, one, or two days a week. The probability that they play one day is 0.1, one day is 0.4 and two days is 0.5. Find the expected value.

Key Terms

- Combinations
- Conditional probability
- Contingency table
- Dependent events
- Event
- Expected value
- Factorial
- Frequency
- Independent events
- Odds in favor
- Odds against
- Probability of an event
- Permutation
- Sample space
- Weight average

Summary

- Factorial

  The factorial function with symbol "!" is the product of all whole numbers from the number you choose down to 1.

- Formula for Factorial

  $n! = n(n-1)(n-2) \ldots 1$

  $n! = n \times (n-1)!$

  $0! = 1.$

- Permutation

  Permutation is an arrangement of objects in a particular order.

  In permutation order is important.

- Permutation formula

  $${}_nP_r \text{ or } P(n, r) = \frac{n!}{(n-r)!};\ r \leq n,$$

  where; n is the total number of objects and r is the number of selected objects arranged in a specific order.

- **Combination**

  Combination is a selection of items from a set that has distinct elements in which the order of selection does not matter.

- Combination formula

  $$\binom{n}{r} = \frac{n!}{r!(n-r)!};\ r \leq n$$

- **Types of Events**

  (i) Simple event
  (ii) Compound event
  (iii) Mutually exclusive events
  (iv) Dependent events
  (v) Independent events

- Conditional probability

  Conditional probability is the likely hood of an event occurring based on the occurrence of a previous event.

- Conditional probability formula

  $$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

  where; $P(A|B)$ represents the probability of occurrence of A given B has occurred.

- Odds in Favor and Odds Against

  The ratio that compares the number of favorable out comes of an event to the number of unfavorable outcomes is said to be odds.

- Formula for odds in favor of an event $= \dfrac{\text{number of unfavorable outcomes}}{\text{number of favorable out comes}}$.

$$= \frac{\text{number of favorable outcomes}}{\text{number of un favorable out comes}}.$$

- Formula for odds against an event
- Expected value

  The expected value of a random variable is the weighted average of its probability distribution.
- Expected value formula

$$E[X] = \sum_{i=1}^{n} x_i P(x_i).$$

Exercises

I. Decide whether each of the following statements are true or false.

1. An outcome is the result of a single trial of an experiment.
2. An event never equals to its sample space.
3. An event is always non-empty set.
4. If two balls are drawn one after the other from a box that contains 3 yellow, 4 blue and 5 white balls without replacement, then the probability that the first is yellow and the second is blue is the same as the probability that the first is blue and the second is yellows.
5. If events A and B are related by $P(A) > P(B)$, then we conclude that event A is more likely to occur than event B.

II. Work outs

6. Three cards are drawn at random from a deck of 52 playing cards without replacement. Find the probability of drawing
   (a) all aces.
   (b) first ace and the next two kings.
   (c) a 6, a king and a 9 in order.
7. In a box there are red and blue marbles. All the same except their colour. 40% of the marbles are red. If two marbles are drawn at random, what is the probability that
   (a) Exactly one is red. (b) Both are red?
8. Find the odds in favor of getting exactly two tails when 3-coins are tossed together.

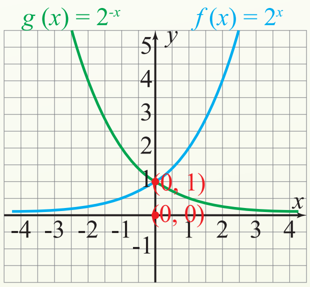
9. By using only five cards numbered 2,3,5,7,9 by placing 3 cards in a row
   (a) How many 3 digit numeral can we form?
   (b) How many of these numbers are even?
   (c) How many of these numbers are less than 500?
10. In a box there are 8 marbles where 4 are black 2 are white and the rest are green. If we draw three marbles at random what is the probability that
    (a) All marbles drawn are black?
    (b) All marbles drawn are of the same colours?
    (c) All marbles are not black?
    (d) All the marbles are of different colours?
11. Two ordinary dice are thrown together and the upper face is observed. Find the probability that
    (a) the two numbers are equal.
    (b) the two numbers are unequal.
    (c) if the two numbers are added, their sum is less than 7.
    (d) The difference of the two numbers is 1.
12. If we toss four coins and observe the sequences of heads and tails, what is the probability to see at least three tails?
13. If we throw a die and observe a number less than 5, what is the probability to see an odd number?
14. The probability that A and B will go to Nazareth for the next Christmas is $\frac{3}{5}$ and $\frac{2}{3}$ respectively. Find the probability that at least one of them will go to Nazreth for the next Christmas.
15. Explain the following statments.
    (a) $E[c] = c$
    (b) $E[X + c) = E[X] + c$
    (c) $E(cX)] = cE[X]$
16. In a clas there are 24 boys and 20 girls. All of the students have either Apple or Sumsung cellphones but not both. If 18 boys and 12 girls have Apple cellphones and 6 boys and 8 girls have Sumsung cellphones, find the probability that a student selected at random.
    (a) has an Apple cellphone, given that the student is a girl.
    (b) is a boy, given that the student has Sumsung cellphone.

M12CH21

CHAPTER 21

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

- evaluate exponential functions;
- graph exponential functions;
- solve application problems involving exponential functions (doubling time growth model, radioactive decay, compound interest);
- distinguish between algebraic and exponential function;
- define base *e*;
- graph exponential functions with base *e*;
- review growth and decay with base *e*;
- solve application problems interest compounding continuously;
- change expressions to logarithmic expressions, and conversely;
- evaluate logarithmic functions;
- evaluate common and natural logarithms using calculator;
- graph logarithmic functions;
- interpret logarithmic functions as inverse of exponential functions;
- determine domain restrictions on logarithmic functions;
- express a single logarithm as a sum or difference of logarithms;
- express a logarithmic expression as a single logarithm;
- evaluate logarithms of a general base other than base 10;
- state the seven basic logarithmic properties;
- state the change one of base formula;
- solve exponential and logarithmic equations.

Introduction

Exponential and logarithmic functions are used to study many naturally occurring phenomenas such as growth of population, decay of radioactive matter, compound interest and depreciation, the pH value of a solution etc.

In this unit we define, discuss and draw the graphs of exponential and logarithmic functions and use them to solve application problems in real life situations.

ACTIVITY 1

1. Let $f(x) = 3^x$. Evaluate

   (a) $f(0)$ (d) $f(1)$ (g) $f\left(\frac{3}{4}\right)$

   (b) $f(-1)$ (e) $f(2)$ (h) $f(0.1)$

   (c) $f(-2)$ (f) $f\left(\frac{1}{2}\right)$ (i) $f(-0.2)$

2. Let $f(x) = 3^x$ and $g(x) = 2^{-x}$. Find

   (a) $(f+g)(0)$

   (b) $(fg)(1)$

   (c) $\left(\frac{f}{g}\right)\left(\frac{-1}{2}\right)$

3. Simplify the following expressions.

   (a) $3^4 \times 2^5$ (c) $\frac{3^3}{3^4}$ (e) $4^2 \times 8^{-3}$ (g) $2^{\frac{2}{3}} \times 4^{\frac{-1}{3}}$

   (b) $\frac{5^6}{5^2}$ (d) $(2^2)^3$ (f) $64^{\frac{-1}{2}} \times 64^{\frac{1}{3}}$

4. What is the difference between the following pairs of function?

   (a) $f(x) = x^2$ and (b) $g(x) = 2^x$

   (b) $f(x) = 2x$ and (b) $g(x) = 2^x$

   (a) $f(x) = \frac{1}{2}x$ and (b) $g(x) = \frac{1}{2^x}$

Exponential rules

let $a > 0$ and $b > 0$ and x and y be real numbers, then it holds true that

1. $a^x a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $a^x b^x = (ab)^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$
6. $a^0 = 1$
7. $a^{-x} = \frac{1}{a^x}$

Algebraic functions

An Algebraic function is a function constructed from polynomials using the algebraic operations addition, subtraction, multiplication, division and taking roots.

Example 1

The following functions are some examples of algebraic functions.

(a) $f(x) = x^2 - 3x + 1$

(b) $g(x) = \sqrt{x}$

(c) $h(x) = \dfrac{3x-2}{5-2x}$

(d) $K(x) = x^3$

The main types of algebraic functions are

- Polynomial functions
- Rational functions
- Power functions

Exponential functions

Functions that can be expressed in the form $f(x) = a^x$, $a > 0$ and $a \neq 1$ are called **exponential functions.** The domain of an exponential function is $(-\infty, \infty)$ and its range is $(0, \infty)$.

Example 2

Each of the following are exponential functions.

$$f(x) = 2^x, f(x) = \left(\frac{1}{2}\right)^x, f(x) = \sqrt{3}^x, f(x) = \pi^x, f(x) = \left(\frac{150}{19}\right)^x$$

Example 3

Let $f(x) = \left(\dfrac{3}{4}\right)^x$

(a) Find x such that $f(x) = \dfrac{64}{27}$

(b) Find $f(x+1) - f(x-1)$

(c) Find $\dfrac{f(2x)}{2f(x)}$

Solution

$f(x) = \left(\dfrac{3}{4}\right)^x$

(a) $f(x) = \dfrac{64}{27}$

$$\Rightarrow \left(\frac{3}{4}\right)^x = \frac{64}{27}$$

$$\Rightarrow \left(\frac{3}{4}\right)^x = \left(\frac{4}{3}\right)^3$$

$$\Rightarrow \left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^{-3} \quad \therefore x = -3.$$

(b) $f(x+1) = \left(\frac{3}{4}\right)^{x+1}$ and $f(x-1) = \left(\frac{3}{4}\right)^{x-1}$

$$\therefore f(x+1) - f(x-1) = \left(\frac{3}{4}\right)^{x+1} - \left(\frac{3}{4}\right)^{x-1}$$

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^x - \left(\frac{3}{4}\right)^{-1}\left(\frac{3}{4}\right)^x$$

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^x - \left(\frac{4}{3}\right)\left(\frac{3}{4}\right)^x$$

$$= \left(\frac{3}{4}\right)^x\left(\frac{3}{4} - \frac{4}{3}\right) = \frac{-7}{12}\left(\frac{3}{4}\right)^x$$

$$= \frac{-7}{12} f(x).$$

(c) $\dfrac{f(2x)}{2f(x)} = \dfrac{\left(\frac{3}{4}\right)^{2x}}{2\left(\frac{3}{4}\right)^x} = \dfrac{1}{2}\left(\dfrac{3}{4}\right)^x.$

Graphs of exponential functions

The exponential functions $f(x) = 2^x$ and $f(x) = \left(\frac{1}{2}\right)^x$

By constructing a table let's see what the graph of $f(x) = 2^x$ or $y = 2^x$ looks like

$y = 2^x$	x	-3	-2	-1	0	1	2	3	4
	y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

We can also see some rational values of x

$y = 2^x$	x	-1.5	0.5	2.5
	y	≈ 0.353553	≈ 1.414214	≈ 5.656854

From the tables we can observe the following

1. $f(x) > 0$ for every value of x
2. $f(0) = 1$, (y-intercept)

3. $f(x) > 1$ iff $x > 0$
4. $0 < f(x) < 1$ iff $x < 0$
5. The value of y goes to zero as x goes to negative infinity.

The x-axis is horizontal asymptote. Therefore, the graph of $f(x) = 2^x$ can be sketched by joining the set of points given in the table. The resulting graph is a smooth curve passing through (0, 1) that approaches the x-axis as x goes to negative infinity and goes far up as x goes to positive infinity.

Properties of $f(x) = 2^x$

1. The domain of f is $\mathbb{R}$
2. It is an increasing function
3. Its range is $(0, \infty)$
4. 2^x approach zero as x approach $-\infty$ and it approach $+\infty$ as x approach $+\infty$.
5. The x-axis is a horizontal asymptote.

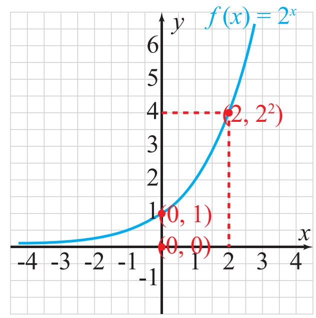


Figure 1

The exponential eunction $y = \left(\frac{1}{2}\right)^x$ or $y = 2^{-x}$

Draw the graph of $f(x) = \left(\frac{1}{2}\right)^x$

We will draw the graph of $f(x) = \left(\frac{1}{2}\right)^x$ based on the following property.

$$g(x) = 2^x \Rightarrow g(-x) = 2^{-x} = \left(\frac{1}{2}\right)^x = f(x).$$

Hence, the graph of $f(x)$ is the image of the graph of $g(x)$ under reflection about the y-axis i.e., the union of the graph of $f(x)$ and $g(x)$ is symmetrical about the y-axis.

Properties of $f(x) = \left(\frac{1}{2}\right)^x$

1. Domain = $\mathbb{R}$
2. Range = $(0, \infty)$
3. It is a decreasing function
4. It approach to zero as x-approach $+\infty$ and approach to $+\infty$ as x-approach $-\infty$.
5. The line $y = 0$ (or the x-axis) is a horizontal asymptote.

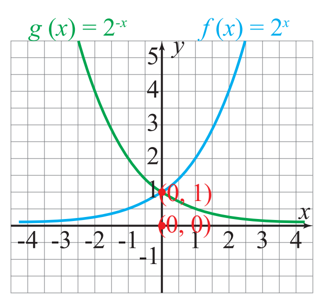


Figure 2.

Remark: From the graph one can see the following properties of $f(x)=\left(\frac{1}{2}\right)^x$

$f(x) > 0$ for every value of x

1. $f(0) = 1$; y – intercept
2. $f(x) > 1$ iff $x < 0$
3. $0 < f(x) < 1$ iff $x > 0$
4. The value of y goes to zero as x goes to positive infinity.

Example 4

Draw the graph of $f(x) = 1 + 2^x$.

Note: To draw the graph of $f(x)$ we first analyze what $f(x)$ means.

$f(x)$ is the sum of two functions which are $g(x) = 1$ and $h(x) = 2^x$.

Said otherwise: $f = \{(x, 1 + 2^x): x \in \mathbb{R}\} = \{(x, 2^x): x \in \mathbb{R}\} + \{(x, 1): x \in \mathbb{R}\}$

Hence, we will draw the graph of $g(x) = 2^x$, and then shift it in the positive y-direction by 1 unit.

Properties of $f(x) = 1 + 2^x$

1. Domain = $\mathbb{R}$
2. Range = $(1, \infty)$
3. It is an increasing function.
4. The graph approach the line $y = 1$.
5. The graph meets the y-axis at (0, 2).

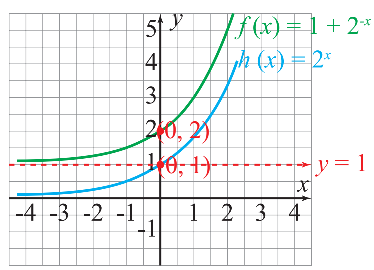


Figure 3.

Properties of Exponential functions

1. Let $f(x) = a^x$; $a > 0$, then
   (i) $f(x + y) = f(x) f(y)$ which is shown as follows.
   $f(x + y) = a^{x+y} = a^x a^y = f(x) f(y)$
   (ii) $f(x-y)=\dfrac{f(x)}{f(y)}$.
   (iii) $f(x^t) = (f(x))^t$.
2.
   - If $a > 1$, then $f(x) = a^x$ is a increasing.
   - If $0 < a < 1$, then $f(x) = a^x$ is a decreasing function.

EXAMPLE 5

Let $f(x) = \frac{1}{2}(a^x + a^{-x})$; $a > 0$. Evaluate (a) $f(0)$. (b) $f(-1)$. (c) $f(1)$.

Solution

(a) $f(0) = \frac{1}{2}\left(a^{o} + a^{-0}\right) = \frac{1}{2}(1+1) = 1.$

(b) $\frac{1}{2}\left(a^{-1} + a^{(-1)}\right) = \frac{1}{2}\left(a + a^{-1}\right).$

(c) $f(1) = \frac{1}{2}\left(a^{1} + a^{-1}\right) = \frac{1}{2}\left(a + a^{-1}\right).$

Let x be any real number, then observe that

$$f(-x) = \frac{1}{2}\left(a^{-x} + a^{-(-x)}\right) = \frac{1}{2}\left(a^{-x} + a^{x}\right) = f(x).$$

The exponential functions $y = a^x$, $y = \left(\frac{1}{a}\right)^x$ or $y = a^{-x}$

Draw the graphs of the $f(x) = a^x$ and $h(x) = \left(\frac{1}{a}\right)^x$ where $a > 0$

The following are the graph of the functions f and h.

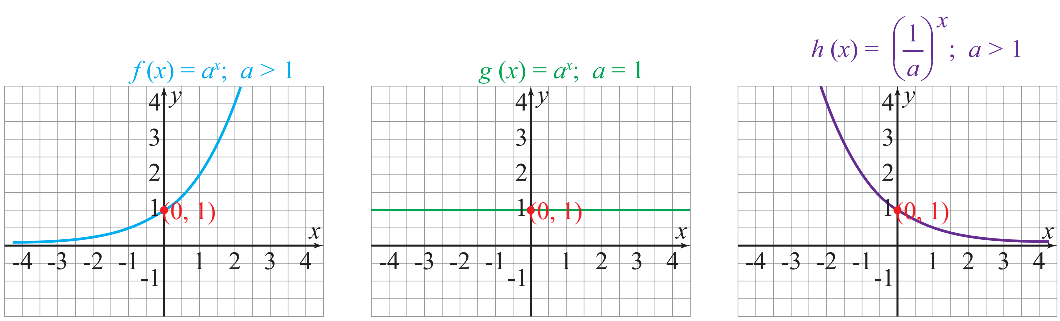


Figure 4.

The graph of h is the image of the graph of f in reflection by the y-axis. When we draw both graphs together we get the graphs in Figure 5.

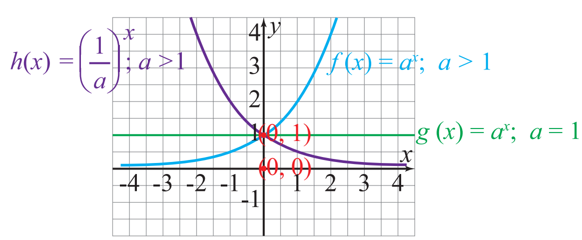


Figure 5.

Properties

1. Domain = $\mathbb{R}$
2. Range = $(0, \infty)$

3. $y = 0$ is horizontal asymptote for $f(x) = a^x$; $a \neq 1$
4. Decreasing for $0 < a < 1$ and increasing for $a > 1$

Example 6

Compare the graph of the functions $f(x) = 2^x$ and $g(x) = 3^x$.

Solution

Look at the following introductory table

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$g(x) = 3^x$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

From the table we see that:

(a) $2^x = 3^x = 1$ iff $x = 0$ (the graphs of $f(x)$ and $g(x)$ meet at (0, 1))

(b) $2^x > 3^x$ iff $x < 0$ (the graph of $f(x)$ is above the graph of $g(x)$ whenever $x < 0$)

(c) $2^x < 3^x$ iff $x > 0$ (the graph of $g(x)$ is above the graph of $f(x)$ whenever $x > 0$). Both functions are smooth and increasing. When $x > 1$, $g(x) = 3^x$ increases more rapidly than $f(x) = 2^x$ and their values are never zero or negative but their graphs approach to the x-axis as x goes to negative infinity.

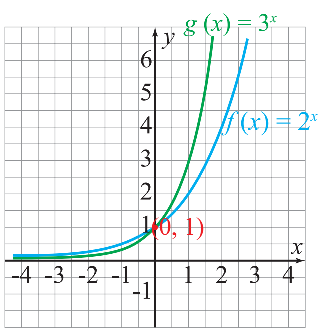


Figure 6.

Example 7

Compare the graphs of $f(x) = a^x$ and $g(x) = b^x$ when $1 < a < b$.

Solution

Observe that:

$1 < a < b$ means that $\left(\frac{b}{a}\right) > 1$

$\Rightarrow \left(\frac{b}{a}\right)^x > 1$ if $x > 0$

$b^x > a^x$ if $x > 0$

(a) The graph of $g(x) = b^x$ is above the graph of $f(x) = a^x$ whenever $x > 0$.

(b) If $x = 0$, then $\left(\frac{b}{a}\right)^0 = 1$.

$\Rightarrow \left(\frac{b}{a}\right)^x = 1$ whenever $x = 0$.

Therefore, the graph of $f(x) = a^x$ and $g(x) = b^x$ meet at (0, 1)

(c) If $x < 0$, then $0 < \left(\frac{b}{a}\right)^x < 1$.

Therefore, the graph of $f(x) = a^x$ is above the graph of $g(x) = b^x$ whenever $x < 0$.

From a, b and c we get the following graphs.

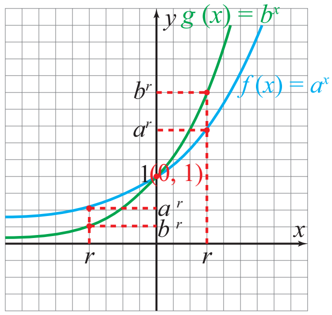


Figure 7.

From the graph we see that

1. If $r > 0$, $b^r > a^r$
2. If $r < 0$, $b^r < a^r$
3. If $r = 0$, $b^r = a^r = 1$

- Both functions are increasing, moreover $g(x) = b^x$ increases more rapidly than $f(x) = a^x$ when $x > 0$.

Algebraic functions

An Algebraic function is a function constructed from polynomials using the algebraic operations addition, subtraction, multiplication, division and taking roots.

Example 8

The following functions are some examples of algebraic functions.

(a) $f(x) = x^2 - 3x + 1$

(b) $g(x) = \sqrt{x}$

(c) $h(x) = \dfrac{3x-2}{5-2x}$

(d) $K(x) = x^3$

The main types of algebraic functions are

- Polynomial functions
- Rational functions
- Power functions

Definition of base *e*

The natural exponential function: The exponential function given by $f(x) = e^x$; where e is an irrational number $e \approx 2.71828182845904\ldots$ is said to be the natural exponential function.

The graph of exponential function with base *e*

The following are graphs of $f(x) = e^x$ as compared to the graphs of $y = 2^x$ and $y = 3^x$ since $2 < e < 3$.

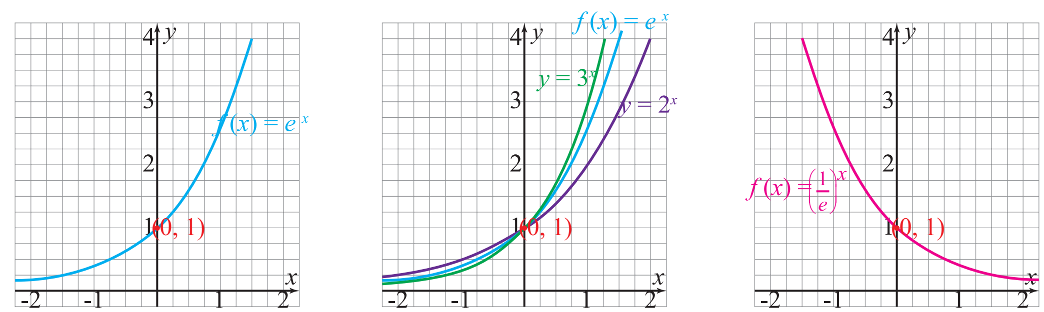


Figure 8. *a.* *b.* *c.*

Exponential equation

- If $a^x = a^y$, for $a > 0$, then $x = y$.

EXAMPLE 9

Solve $3^x = 1$

Solution

$3^x = 1$

$\Rightarrow 3^x = 3^0$

$\Rightarrow x = 0$

$\therefore$ S.S $= \{0\}$.

EXAMPLE 11

Solve $2^x = 64$

Solution

$2^x = 64$

$\Rightarrow 2^x = 2^6$

$\therefore$ S.S $= \{6\}$.

EXAMPLE 10

Solve $\left(\dfrac{1}{8}\right)^{x+5} = 4$

Solution

$\left(\dfrac{1}{8}\right)^{x+5} = 4$

$\Rightarrow (2^{-3})^{x+5} = 2^2$

$\Rightarrow 2^{-3x-15} = 2^2$

$\Rightarrow -3x - 15 = 2$

$\Rightarrow -3x = 17$

$\Rightarrow x = \dfrac{-17}{3}$

$\therefore$ S.S $= \left\{\dfrac{-17}{3}\right\}$.

Example 12

Solve $25x - \frac{1}{2} = 5^{x+2}$

Solution

$(25)^{x-\frac{1}{2}} = 5^{x+2}$

$\Rightarrow (5^2)^{x-\frac{1}{2}} = 5^{x+2}$

$\Rightarrow 5^{2x-1} = 5^{x+2}$

$\Rightarrow 2x - 1$

$\therefore$ S.S = {3}.

Example 13

$\sqrt{2^x} = 32$

$\sqrt{2^x} = 2^5$

$\Rightarrow 2^{\frac{1}{2}x} = 2^5$

$\Rightarrow \frac{1}{2}x = 5 \Rightarrow x = 10$

$\therefore$ S.S = {10}.

Example 14

Solve $5^{\sqrt{x}} = 125$

Solution

$5^{\sqrt{x}} = 125$

$\Rightarrow 5^{\sqrt{x}} = 5^3$

$\Rightarrow \sqrt{x} = 3$

$\Rightarrow x = 9$

$\therefore$S.S = {9}.

Example 15

Solve $2^{3x} = 8^{3-2x}$

Solution

$2^{3x} = 8^{3-2x}$

$= (2^3)^{(3-2x)}$

$= 2^{9-6x}$

$\Rightarrow 3x = 9 - 6x$

$\Rightarrow 9x = 9$

$\Rightarrow x = 1$

$\therefore$S.S = {1}.

Exercises

I. True or False Items

1. $(-1)^{-1} = 0.1$
2. $(-1)^0 = 1$
3. $(2)^{3x} = 6^x_1$
4. $2^{\sqrt{2}} = 4^{\sqrt{2}}$
5. $\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2 = a + b$ for $a \geq 0, b \geq 0$
6. $\sqrt[5]{x} \times \sqrt[5]{x^3} = x^{0.8}$
7. $\left(x^2y^{\sqrt{2}}\right)^2 = x^4y^2$
8. $5 \times 3^x = (15)^x$
9. $\left(a^{x-1}\right)^{-1} = a^{\frac{1}{x-1}}$
10. $\left(2+\sqrt{2}\right)^{\frac{1}{3}}\left(2-\sqrt{2}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}$

II. Simplify each of the following

11. $\left(5^3\right)^2 \times \left(\frac{2}{5}\right)^2$
12. $\frac{2^5}{2^3}$
13. $\frac{3^5 \times 2^2}{\left(6^3 \div 3^2\right)}$

14. $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

15. $\dfrac{(6x)^{\frac{1}{2}}}{\sqrt{6x^{\frac{3}{2}}}}$

16. $\dfrac{\sqrt{0.0064}}{\sqrt{16900}}$

17. $\sqrt[6]{8^2}$

18. $\left(\dfrac{25}{27}\right)^{\frac{-3}{2}} \times \left(\dfrac{3}{4}\right)^{1.5}$

19. $\dfrac{\left(5^2\right)^{\frac{-1}{2}} \left(6^3\right)^{-1}}{(8)^{-3}}$

20. $\left(\dfrac{x^{-8}y^{-2}}{y^{-3}x^{-5}}\right)^{-5}$

21. $\left(\dfrac{8\left(a^{\frac{3}{4}}\right)y}{5a^{\frac{1}{2}}b}\right)^2$

22. $\dfrac{(x^3y^2)^4\, z^4t^5}{(x^2y^3)^3(z^3t^2)^4}$

23. $\dfrac{(x+1)(x-1)^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}}{x}$

24. $\dfrac{\left(a^{\sqrt{x}} . a^{\frac{1}{\sqrt{x}}}\right)^x}{\left(a^{x+1}\right)^{\sqrt{x}}}$ when $a > 0$

III. Solve each of the following Exponential equations.

25. $2^x = 16$

26. $3^{x+1} = 27^{1+2x}$

27. $9^{3x+1} = \dfrac{1}{81}$

28. $\left(\dfrac{1}{5}\right)^{2x+5} = 125$

29. $(0.4^{x+1})(2.5)^{3-2x} = \dfrac{125}{8}$

30. $5^{x+1} = 5^{3x-2}$

31. $3^x = 729$

32. $9^{x-\frac{1}{2}} - 3^{x-1} - 2 = 0$

33. $(0.1)^{x+5} = 100^{(4-2x)}$

IV. Write each of the following in a single exponent and base, if possible.

34. $x^{n-1} . x^{n+1}$

35. $\dfrac{x^n y^m}{xy}$

36. $\dfrac{x^{m-n} y^{n-m}}{x^{m+n} y^{n+m}}$

37. $2^n + 2^n + 2^n + 2^n$

38. $2^{-n} + 2^{1-n} + 2^{2-n}$

39. $\dfrac{(x^m)^{2n+1}}{(x^{2n+1})^m}$

40. $2^{4-n} - 2^{3-n}$

V. Draw the graph of the following functions (Questions 41–48)

41. $f(x) = 5^x$

42. $f(x) = 2 + 3^x$

43. $f(x) = 4^x - 9$

44. $f(x) = 6 - 3(2)^x$

45. $f(x) = 0.25^x$

46. $f(x) = \left(\frac{3}{4}\right)^x - 2$

47. $f(x) = 5 - \left(\frac{3}{2}\right)^x$

48. $f(x) = -\left(\frac{1}{2}\right)^x$

Application problems on exponential functions

Mathematical model

A mathematical model is an equation or a set of equations in which variables represent real world quantities.

Compound interest, population growth, and radioactive decay are applications of exponential functions.

ACTIVITY 2

1. The cost of a new car is L$ 40,000. Every year the car losses 10% of its previous year value.
   Fill in the following table.

Time (years)	10% (loss)	90% value
0	0	L$ 40,000
1	$10\% \times 40{,}000 = \$4000$	L$ (40, 000 – 4000) = L$ 36000
2	$10\% \times 36000 = 3600$	L$ (36000 – 3600) = L$ 32,400
3		
4		
5		

2. In a laboratory, under favorable conditions, a growing bacterial population doubles every hour. If there is 250c.c bacteria in the laboratory, what will be its size after 2 hours?

Compound interest

Compound interest (or compounding interest) is the interest on a loan base or deposite calculated based on both initial principal and accumulated interest from previous periods.

Compound Interest Formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = Final Amount

P = Initial Principal Balance

r = Interest Rate

n = Number of Times Interest applied per Time Period.

t = Number of Time Periods Elapsed.

EXAMPLE 16

A woman invested L\$ 20,000 in a bank for 10 years. Find the amount of money she will have if the bank pays 12% interest per annum compounded.

(a) annually
(b) Semi - annually
(c) quarterly
(d) monthly
(e) weekly
(f) daily

Solution

Using the formula and a calculator we have,

(a) $n = 1$

Hence $A = 20000\left(1+\frac{0.12}{1}\right)^{10\times 1}$

$= 20,000\,(1.12)^{10}$

$A =$ L\$ 62,116.96

$A = P + I$
$I = A - P$
$I =$ L\$ 42,116.96

(b) $n = 2$

$A = 20,000\left(1+\frac{0.12}{2}\right)^{10\times 2}$

$= 20,000\,(1.06)^{20}$

$A =$ L\$ 64,142.71

$I =$ L\$ 44,142.71

(c) $n = 4$

$A = 20,000\left(1+\frac{0.12}{4}\right)^{10\times 4}$

$= 20,000\,(1.03)^{40}$

$A =$ L\$ 65,240.76

$I =$ L\$ 45,240.76

(d) $n = 12$

$A = 20,000\left(1+\frac{0.12}{12}\right)^{10\times 12}$

$= 20,000\,(1.01)^{120}$

$A =$ L\$ 66,007.74

$I =$ L\$ 46,007.74

(e) $n = 52$

$A = 20{,}000\left(1 + \dfrac{0.12}{52}\right)^{10 \times 52}$

$= 20{,}000\,(1.0023)^{520}$

$A = \text{L\$ } 66{,}310.60$

$I = \text{L\$ } 46{,}310.60$

(f) $n = 365$

$A = 20{,}000\left(1 + \dfrac{0.12}{365}\right)^{10 \times 365}$

$= 20{,}000\,(1.000328)^{3650}$

$A = \text{L\$ } 66{,}389.24$

$I = \text{L\$ } 46{,}389.24$

Continuous compounding

Activity 3

Suppose you invest L$ 1000 in a bank that pays 5% annual interest for one year compounded daily, hourly, perminuits, fractions of minutes and so on, does the money has an ending vlaue?

What is the maximum money you may have in the bank after one year.

Continuous compounding is the limit that compound interest can reach if it is calcualted and invested an account's balance over infinite period of times.

When the number of terms is infinite, as $n \to \infty$, $\left(1 + \dfrac{1}{n}\right)^n \to e$

Thus, the continuous compounding formula is $A = Pe^{rt}$.

Example 17

A girl invested L$ 1000 in a bank that pays an annual interest of 7% compounded continuously. What is the amount she can get after 12 years from the bank?

Solution

$P = \text{L\$}1000$, $r = 7\% = 0.07$, $t = 12$ years

Substituting these values in the formula

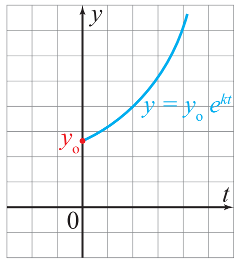
$A = Pe^{rt}$ gives

$A = \text{L\$ } 1000\, e^{0.07 \times 12}$

$= \text{L\$ } 2{,}316.37$

Exponential growth and decay

Using exponents and logarithms, certain physical quantities can be described how they vary with respect to time t. We say y increases or grows exponentially as a function of t if there are positive constants y_0 and k such that



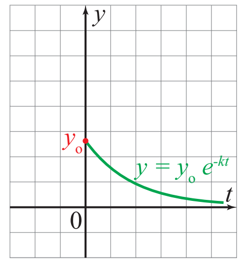


Figure 9. a. b.

$$y = y_0 e^{kt}$$

If $y = y_0\, e^{-kt}$, we say that y decreases or decays exponentially, y is the amount of the element in the sample t – time after the measurements started.

If $t = 0$, $y = y_0\, e^{k(0)} = y_0$ and $y = y_0 e^{-k(0)} = y_0$.

$\Rightarrow y_0$ is the amount of the substance at time $t = 0$

Money in a saving account with continuous compound interest grows exponentially y_0 = the principal and k = annual interest rate. Radioactive elements decay exponentially.

Example 18

In a given colony there are 100 rabbits. If the colony grows to 150 after 1 month, how many rabbits will be in this colony after 6 months?

Solution

$y = y_o e^{kt}$

$t = 0 \quad \Rightarrow y_o = 100.$

$\Rightarrow y = 100e^{kt}$

$t = 1$ month $\quad \Rightarrow y = 150$

$\Rightarrow 150 = 100e^{k(1)}$

$$\Rightarrow \frac{150}{100} = e^k \Rightarrow e^k = \frac{3}{2}$$

$$\Rightarrow y = 100 \times \left(\frac{3}{2}\right)^t$$

$$t = 6 \text{ months} \Rightarrow y = 100 \times \left(\frac{3}{2}\right)^6$$

$y \approx 1139$ rabbits.

Example 19

In a given experiment it is found that the number of bacteria N in a culture after t days is given by $N = 1000 \times 4^{3t}$. Find the number of bacteria.

(a) at the beginning (b) after $5\frac{1}{2}$ days

Solution

(a) The number of bacteria at the beginning or at $t = 0$ is $1000 \times 4^{3 \times 0} = 1000$.

(b) The number of bacteria after $5\frac{1}{2}$ days is

$N = 1000 \times 4^{3 \times 5.5} = 1000 \times 4^{16.5}$

$= 1000 \times 8589934592$

$= 8589934592000$ bacteria

Note also that $4^{16.5} = (2^2)^{16.5} = 2^{33}$.

Half - Life

If a given element decays exponentially, the amount of time that it takes to decrease by 50% is called the half – life

$$y = y_o e^{kt}$$

$$y = \frac{1}{2} y_o \Rightarrow \frac{1}{2} y_o = e^{kt}$$

$$\Rightarrow \frac{1}{2} = e^{kt} \Rightarrow e^k = \left(\frac{1}{2}\right)^{\frac{1}{t}}.$$

Example 20

What is the half-life of the radioactive material plutonium 239, if the material decays according to the function $y = y_o\left(\frac{-t}{2^{24,400}}\right)$, where y_o is the original mass?

Solution

In half-life y will be equal to $\frac{1}{2} y_o$

$$\frac{1}{2} y_o = y_o\left(\frac{-t}{2^{24,400}}\right)$$

$$\Rightarrow \frac{1}{2} = 2^{\frac{-t}{24,400}}$$

$$\Rightarrow \frac{1}{2} = \left(\frac{1}{2}\right)^{\frac{t}{24,400}}$$

$$\Rightarrow 1 = \frac{t}{24,400}$$

t = 24,400. (This is half life time)

EXERCISES

1. A person invests L$ 10,000 in a saving account with 5% annual interest. Find the amount after 6 years if the interest is compounded.
   (a) annually
   (b) quarterly
   (c) semi - annually
   (d) monthly
2. A woman invests L$ 3,425 in a retirement account with a fixed annual interest rate of 11% compounded continuously. What will be her account balance after 12 years?
3. The population of a certain city increases 2.5% per year. If its current population is 16,570 what will the population be 7 years from now?
4. The population of bacteria in a culture increases 2% per minute, If the current population is 192 million, find its population after 8 minutes.
5. The decay rate of a radioactive isotope element is 12.10% per thousand years and the current mass is 148.4mg. Find its mass after
   (a) 2.4 thousand years.
   (b) 5.72 thousand years.

Introduction

Logarithmic function is an important notion of function in the context of solving problems, analyzing graphs and computing inverses. They are useful in grasping and visualizing equations and inequalities and also in determining the range and the domain of functions.

Definitions of logarithms

In the expression $y = a^x$, $a > 0$ and $a \neq 1$, $x > 0$; we need to express x in terms of y. If the graph of an exponential function, $f(x) = a^x$; $a > 0$ and $a \neq 1$, is reflected by the line $y = x$, then it gives the graph of a function called logarithmic function given by $y = \log_a x$.

The following are the graphs of logarithmic functions.

Case 2: $a > 1$

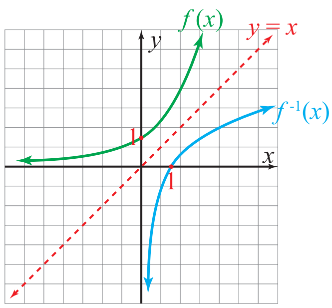


Figure 10.

Case 2: $0 < a < 1$

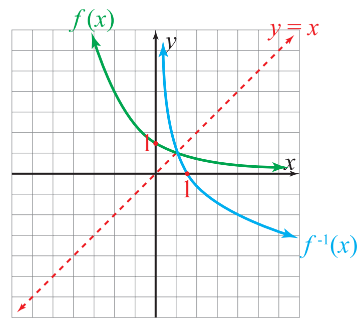


Figure 11.

In case 2 the graphs meet at the point they cross the line $y = x$.

Definition

Let $a \in \mathbb{R}$ such that $a > 0$ and $a \neq 1$. The function: $f(x) = \log_a x$ is called Logarithmic Function.

Note: Domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.

Read "$\log_a x$" as: logarithm of x to the base a. Or, log x to the base a.

$y = \log_a x$ if and only if $x = a^y$.

Example 21

(a) $\log_2 1 = 0$ since $2^0 = 1$

(b) $\log_2 4 = 2$ since $2^2 = 4$

(c) $\log_3 81 = 4$, since $3^4 = 81$

(d) $\log_2 8 = 3$, since $2^3 = 8$

(e) $\log_2 \left(\frac{1}{8}\right) = -3$, since $2^{-3} = \frac{1}{8}$

(f) $\log_8 \left(\frac{1}{64}\right)^2 = -4$ since $8^{-4} = \left(\frac{1}{64}\right)^2$

(g) $\log_1 8 =$ does not exist, since $1^x = 8$ has no solution

By the definition, the base of a logarithm is not equal to 1.

Example 22

Converting equations from logarithmic form to exponential form.

Logarithmic form	Exponential form
$2 = \log_2 4$	$2^2 = 4$
$3 = \log_2 8$	$2^3 = 8$
$-1 = \log_5 \frac{1}{5}$	$5^{-1} = \frac{1}{5}$
$-1 = \log_{\left(\frac{1}{3}\right)} 3$	$\left(\frac{1}{3}\right)^{-1} = 3$
$0 = \log_\pi 1$	$\pi^0 = 1$
$4 = \log_{\sqrt{3}} 9$	$\sqrt{3}^4 = 9$
$y = \log_a x$; $a > 0, a \neq 1$	$a^y = x$

Example 23

Simplify the following

(a) $\log_2 32$ (b) $\log_{\sqrt[5]{125}} \sqrt[3]{5^2}$

Solution

(a) Let $y = \log_2 32$, then $2^y = 32$ and $y = 5$.

(b) Let $h = \log_{\sqrt[5]{125}} \sqrt[3]{5^2}$, then $\sqrt[3]{5^2} = (\sqrt[5]{125})^h$

$$5^{\frac{2}{3}} = 5^{\frac{3h}{5}} \Rightarrow \frac{2}{3} = \frac{3h}{5}$$

$$\Rightarrow h = \frac{10}{9}$$

Example 24

Solve $\log_8 x = \frac{1}{3}$

Solution

$$\log_8 x = \frac{1}{3} \Rightarrow x = 8^{\frac{1}{3}}$$

$$\Rightarrow x = \left(2^3\right)^{\frac{1}{3}} \Rightarrow x = 2$$

$\therefore$ S.S = {2}.

Exercises

I. Rewrite each logarithmic equation as an equivalent exponential equation (Question 1−10)

1. $\log 2^4 = 2$
2. $\log_8 \dfrac{1}{2} = \dfrac{-1}{3}$
3. $\log \dfrac{25}{\sqrt{5}} = 4$
4. $\log_{\pi}^{\pi} = 1$
5. $\log_7 (7^y) = y$
6. $\log_m (m^3) = 3$
7. $\log_7 49^{(-0.5)} = -1$
8. $\log_2 \sqrt[5]{4} = \dfrac{5}{2}$
9. $\log_{125} 1 = 0$
10. $\log (1.25) = 0.0969$

II. Simplify the following (Questions 11–20)

11. $\log_2 64$
12. $\log_{\frac{1}{2}} 8$
13. $\log_{\sqrt[7]{16}} \left(\dfrac{1}{128}\right)$
14. $\log_{64} 32$
15. $\log_{\left(\frac{1}{8}\right)} 4$
16. $\log_{\sqrt[5]{\left(\frac{3}{2}\right)}} \left(\dfrac{\sqrt[5]{8}}{\sqrt[10]{729}}\right)$
17. $\log_{\left(\frac{3}{5}\right)} \left(\dfrac{125}{27}\right)$
18. $\log_{\sqrt[3]{81}} \left(\dfrac{1}{27}\right)$
19. $\log_{\sqrt[3]{16}} \sqrt[5]{8}$
20. $\log_{\sqrt[3]{16}} 8^{\frac{1}{4}}$

III. Solve the following (Questions 21–26)

21. $\log_2 x = 0.25$
22. $\log_3 3x - 1 = 2$
23. $\log_x x = 2$
24. $\log_2 x^2 - 1 = 3$
25. $\log_2 x - 6 = 2$
26. $\log_3 x^2 + 1 = 2$

Graphs of logarithmic functions

Example 25

Draw the graph of $f(x) = \log_2 x$.

Solution

Note that $f(x) = \log_2 x$ is the inverse of the exponential function $g(x) = 2^x$. The following tables show this relationship.

x	−3	−2	−1	0	1	2	3	4
$f(x) = \log_2 x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$f(x) = \log_2 x$	−3	−2	−1	0	1	2	3	4

Note: If (a, b) is in f, then (b, a) is in f^{-1}.
For instance $(4, 16) \in \{(x, y): y = 2^x\}$ and $(16, 4) \in \{(x, y): y = \log_2 x\}$.
Hence, reflecting the graph of $g(x) = 2^x$, in the line $y = x$ gives the graph of $f(x) = \log_2 x$.

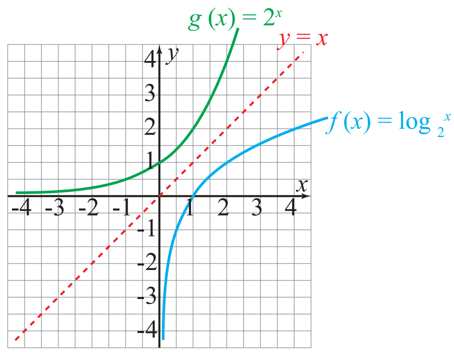


Figure 12.

$g(x) = 2^x$	$f(x) = \log_2 x$
Domain = $\mathbb{R}$	Domain = $(0, \infty)$
Range = $(0, \infty)$	Range = $(-\infty, \infty)$
Increasing	Increasing
one-to-one	one-to-one
No x-intercept	$(1, 0)$, x-intercept
$(0, 1)$, y-intercept	No y-intercept
Do not meet $y = x$	Do not meet $y = x$
Horizontal asymptote, x-axis	Vertical asymptote, y-axis

Example 26

Draw the graph of $f(x) = \log_{\left(\frac{1}{2}\right)} x$

Solution

$f(x) = \log_{\left(\frac{1}{2}\right)} x$ is the inverse of the exponential function

$g(x) = \left(\frac{1}{2}\right)^x$, hence, reflecting the graph of g in the line $y = x$ gives the graph of f.

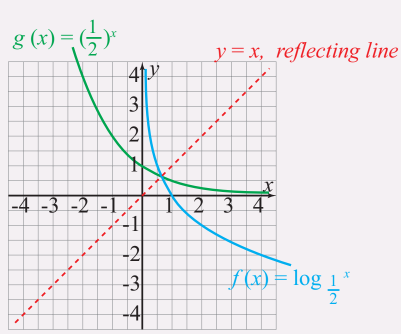
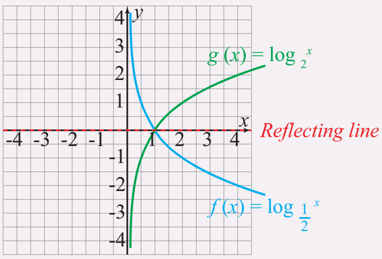


Figure 13.

$g(x) = \left(\frac{1}{2}\right)^x$	$f(x) = \log_{\frac{1}{2}} x$
Domain, $(-\infty, \infty)$	Domain, $(0, \infty)$
Range, $(0, \infty)$	Range, $(-\infty, \infty)$
Decreasing	Decreasing
one-to-one	one-to-one
$(0, 1)$, y-intercept	$(1, 0)$, x-intercept
meet $y = x$	meet $y = x$
Horizontal asymptote, x-axis	Vertical asymptote, y-axis



$\log_{\frac{1}{2}} x = -\log_2 x$. Hence reflecting the graph of

$g(x) = \log_2 x$ in the x-axis gives the graph of

$-g(x) = -\log_2 x = \log_{\left(\frac{1}{2}\right)} x = f(x)$

Figure 14.

$g(x) = \log_2 x$	$-g(x) = f(x) = \log_{\left(\frac{1}{2}\right)} x$
Domain, $(0, \infty)$	Domain, $(0, \infty)$
Range, $\mathbb{R}$	Range, $\mathbb{R}$
Increasing	Decreasing
$(1, 0)$, x-intercept	$(1, 0)$, x-intercept
Asymptote, y-axis	Asymptote, y-axis
Positive on $(1, \infty)$	Positive on $(0, 1)$
Negative on $(0, 1)$	Negative on $(1, \infty)$

Note: If $0 < a < 1$, then the general shape of the graph of $f(x) = \log_a x$ is shown below.

- Domain $= (0, \infty)$
- Range $= \mathbb{R}$
- Vertical Asymptote y-axis
- Decreasing

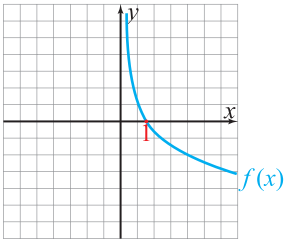


Figure 15.

- $f(x) > 0$ whenever $0 < x < 1$
- $f(x) = 0$, when $x = 1$
- $f(x) < 0$, when $x > 1$

If $a > 1$, then the general shape of the graph of $f(x) = \log_a x$ is shown below.

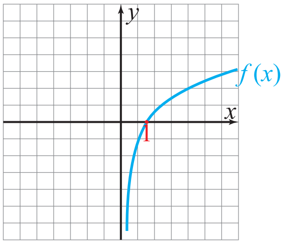


Figure 16.

- Domain = $(0, \infty)$
- Range = $\mathbb{R}$
- Vertical Asymptote y-axis
- Increasing
- $f(x) > 0$, when $x > 1$
- $f(x) = 0$, when $x = 1$
- $f(x) < 0$, when $0 < x < 1$

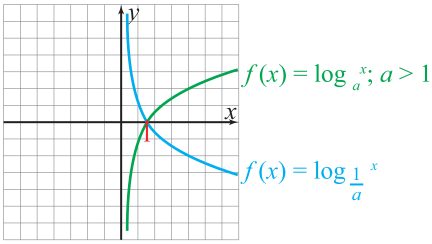


Figure 17.

EXAMPLE 27

Draw the graph of each of the following function on the same coordinate plane.

(a) $f(x) = \log_3 x$

(b) $g(x) = \frac{1}{2}\log_2 x$

(c) $h(x) = \log_5 x$

(d) $R(x) = \log_7 x$

Solution

Note that in b, $f(x) = \frac{1}{2}\log_2 x = \log_{2^2} x = \log_4 x$

Also, $1 < 3 < 4 < 5 < 7$, then

(i) For $x > 1$, $\log_3 x > \log_4 x > \log_5 x > \log_7 x > 0$

(ii) For $x = 1$, $\log_3 1 = \log_4 1 = \log_5 1 = \log_7 1 = 0$

(iii) For $0 < x < 1$, $\log_3 x < \log_4 x < \log_5 x < \log_7 x < 0$

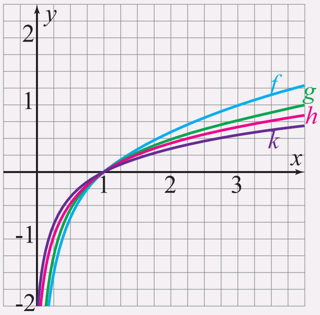


Figure 18.

EXAMPLE 28

Draw the graphs of each of the following functions on the same xy-plane.

(a) $f(x) = \log_{0.1} x$

(b) $g(x) = \log_{\left(\frac{1}{3}\right)} x$

(c) $h(x) = -\log_5 x$

(d) $k(x) = \log_6 \frac{1}{x}$

Solution

Note that $h(x) = -\log_5 x = \log_{\left(\frac{1}{5}\right)} x$ and $k(x) = \log_6 \frac{1}{x}$

$= -\log_6 x = \log_{\frac{1}{6}} x$

Also, $0.1 < \frac{1}{6} < \frac{1}{5} < \frac{1}{3}$, hence

(i) For $x > 1$, $\log_{0.1} x > \log_{\frac{1}{6}} x > \log_{\frac{1}{5}} x > \log_{\frac{1}{3}} x$

(ii) For $x = 1$, $\log_{0.1} 1 = \log_{\frac{1}{6}} 1 = \log_{\frac{1}{5}} 1 = \log_{\frac{1}{3}} 1 = 0$

(iii) For $0 < x < 1$, $\log_{0.1} x < \log_{\frac{1}{6}} x < \log_{\frac{1}{5}} x < \log_{\frac{1}{3}} x$

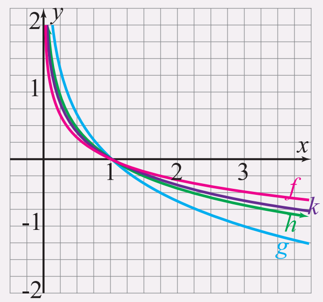


Figure 19.

Exercises

1. Draw the graphs of $y = \log_{\left(\frac{1}{2}\right)} x$ and $y = \log_{\left(\frac{1}{3}\right)} x$ in the same coordinate plane, then fill in the following blank spaces.

   (a) $\log_{\left(\frac{1}{2}\right)} x > \log_{\left(\frac{1}{2}\right)} x$ for the value of x where ______________________

   (b) $\log_{\left(\frac{1}{2}\right)} x < \log_{\left(\frac{1}{3}\right)} x$ for the value of x where ______________________

Draw the graph of each of the following functions. Determine the domain and the range. Find the x or y-intercept and the vertical or horizontal asymptote. Indicate whether it is increasing or decreasing.

2. $f(x) = \log_2(x - 2)$
3. $f(x) = 4 - \log_{\left(-\right)}(\quad 5)$
4. $f(x) = -2 + \log_5(2x + 1)$
5. $f(x) = \log_7 x^2$
6. $f(x) = \log_{\left(\frac{1}{3}\right)} \sqrt{x}$
7. $f(x) = \log_{0.25}(-x)$
8. $f(x) = \log_{\frac{1}{2}}(3 - 2x)$
9. $f(x) = \log_2(1 - x)$
10. $f(x) = 3 + \log_2(4x - 1)$

Theorem on logarithms

To simplify logarithmic expressions and solve logarithmic equations the following theorems are very important.

Theorem : Let x, y, $a > 0$ and $a \neq 1$, $k \in \mathbb{R}$. Then

1. $\log_a xy = \log_a x + \log_a y$ (Product rule for logarithms)
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$ (Quotient Rule for logarithms)
3. If $y \neq 1$, then $\log_y x = \frac{\log_c x}{\log_c y}$, for all $c > 0$ and $c^{\prime}1$ (change of Base)
4. $\log_a y^k = k \log_a y$ (Power rule of logarithms)
5. If $k \neq 0$, $\log_{(a^k)} x = \frac{1}{k} \log_a x$.
6. $\log_a x = x$ (The identify)

Simplifications of logarithmic expressions and logarithmic equations

- Now apply the above theorems to simplify logarithmic expressions and solve logarithmic equations. In any logarithmic equation, $\log_a x = 0$ implies $x = 1$.

EXAMPLE 29

Simplify the following logarithms using the above theorem.

(a) $\log_{128} 32$ (b) $\log_{\sqrt[4]{27^5}} \sqrt[3]{81^5}$

Solution

(a) $\log_{128} 32 = \log_{2^7} 2^5 = 5\log_{2^7} 2 = 5\log_2 2 = \frac{5}{7}$.

(b) $\log_{\sqrt[4]{27^5}} \sqrt[3]{81^5} = \log_{3^{\frac{15}{4}}} 3^{\frac{20}{3}} = \left(\frac{20}{3}\right)\frac{(4)}{15}\log_3 3 = \left(\frac{20}{3}\right)\left(\frac{4}{15}\right) = \frac{16}{9}$.

EXAMPLE 30

Solve the following logarithmic equations.

$\log_2 2(x+1) + \log_2 (x+1) = 5; x > -1$

Solution

$\log_2 2(x+1) + \log_2 (x+1) = 5;\ x > -1$

$\Rightarrow \log_2 2 + \log_2 (x+1) + \log_2 (x+1) = 5 \Rightarrow 1 + 2\log_2 (x+1) = 5$

$\Rightarrow 2\log(x+1) = 4 \Rightarrow \log_2 (x+1) = 2 \Rightarrow (x+1) = 2^2 = 4 \Rightarrow x = 4 - 1 = 3$

$\therefore$ S.S = $\{3\}$.

Example 31

Let $\log_a 3 = y$ and $\log_a 2 = k$, for $a > 1$, Simplify

(a) $\log_a 6$ (b) $\log_a \frac{81}{64}$ (c) $\log_8 9$

Solution

(a) $\log_a 6 = \log_a 2 \times 3 = \log_a 3 + \log_a 2 = y + k$.

(b) $\log_a \frac{81}{64} = \log_a 81 - \log_a 64 = 4\log_a 3 - 6\log_a 2 = 4y - 6k$.

(c) $\log_8 9 = \frac{\log_a 9}{\log_a 8} = \frac{2y}{3k}$.

Example 32

If $f(\log_9 x) = x$, then which of the following is true?

(a) $f(x) = \log_9 x$

(b) $f(x) = 9^x$

(c) $f\left(\frac{1}{2}\right) = (27)$

(d) $f(0) = 0$

Solution

Let $y = \log_9 x$ then $x = 9^y$

$f(\log_9 x) = x \Rightarrow f(y) = 9^y \Rightarrow f(x) = 9^x$

$\Rightarrow f\left(\frac{1}{2}\right) = 9^{\frac{1}{2}} = 3$

$f(0) = 9^0 = 1$.

Answer B

Example 33

Solve $\log_3(x-1) = 0$

Solution

$\log_3(x-1) = 0 \Rightarrow x - 1 = 3^0$

$\Rightarrow x = 2$

$\therefore$ S.S = $\{2\}$.

Example 34

Solve $\log_{0.2}(1-2x)=1$

Solution

$\log_{0.2}(1-2x)=1$. The universe is $U=\{x:1-2x>0\}=\left\{x:x<\frac{1}{2}\right\}$

$\Rightarrow 1-2x=(0.2)^1$ and $x<\frac{1}{2}$

$\Rightarrow 1-2x=0.2$

$\Rightarrow 1-0.2=2x$

$\Rightarrow 0.8=2x$

$\Rightarrow 0.4=x$

$\therefore$ S.S = {0. 4}.

Example 35

Solve $\log_9(2x+1)-\log_9(7x-1)=-0.5$

Solution

$\log_9(2x+1)-\log_9(7x-1)=-0.5$; $2x+1>0$ and $7x-1>0 \Rightarrow U=\left(\frac{1}{7},\infty\right)$

$\Rightarrow \log_9\left(\frac{2x+1}{7x-1}\right)=-0.5 \Rightarrow \frac{2x+1}{7x-1}=9^{-0.5} \Rightarrow \frac{2x+1}{7x-1}=\frac{1}{3} \Rightarrow \frac{2x+1}{7x-1}-\frac{1}{3}=0$

$\Rightarrow \frac{6x+3-7x+1}{7x-1}=0 \Rightarrow \frac{4-x}{7x-1}=0$

$\Rightarrow x=4$

$\therefore$ S.S = {4}.

Exercises

I. Solve the following logarithmic equations (Questions 1–12)

1. $\log_2 x=5$
2. $\log_x 54=3$
3. $\log_2(x-1)=4$
4. $\log_2(5x-1)=\log_2(8x-13)$
5. $5\log_5 71+3\log_3 15=6\log_6 x$
6. $2^{4+\log_2 x}=1$
7. $3^{(1-\log_9 x)}=\frac{3}{2}$
8. $2^{(4+\log_2 x)}=4$
9. $\log_3(2x+3)+\log_3 x=3$
10. $2\log_5 5+\log_{25}(4x+5)=3$
11. $\log_4(3x-2)-\log_2(5x+2)=-3$

12. $\log_5 x - \log_x 5 = 1.5$
13. Write each of the following in a single base.
    (a) $\log_2 3 + \log_2 x$ (b) $\log_4 7 - \log_2 x$
    (c) $2 \log x + 3\log (x - 1) - \log 2$
    (d) $\log\left(x + \frac{x}{y}\right) - \log\left(k + \frac{k}{y}\right)$
14. If $y = 10^{x^2 - x}$, express x-interms of y.
15. If $y = \frac{a^x - a^{-x}}{a^x + a^x}$; $a > 0$ and $a \neq 1$, express x-interms of y.
16. Let $f(x) = \log (2x - 1)$. Solve $f(x) = \frac{1}{2}$.

Logarithms in base -10 (Common logarithms)

Standard notation

When we write a number in standard notation we will write the number as a product of two numbers x_1 and 10^n where $1 \leq x_1 < 10$ and $n \in \mathbb{Z}$.

EXAMPLE 36

Write the following numbers instandard notation.

(a) 257 (b) 0.451 (c) 25793 (d) 0.0002156

Solution

(a) $257 = 2.57 \times 10^2$ where $x_1 = 2.57$ and $n = 2$ and $n \in \mathbb{Z}$.

(b) $0.451 = 4.51 \times 10^{-1}$.

(c) $25793 = 2.5793 \times 10^4$.

(d) $0.0002156 = 2.156 \times 10^{-4}$.

DEFINITION

Evaluate the following logarithms

(a) $\log_{10} 10$? (c) $\log_{10} 100$ (e) $\log_{0.1} 100$
(b) $\log_{10} 1$? (d) $\log_{10} 0.01$

Notation: Logarithms to the base 10 are used frequently in practice. It has a special name called common logarithms and also it has a special notation.

Let $x > 0$, then $\log_{10} x$ is denoted by $\log x$ such as $\log_{10} 5 = \log 5$.

Example 37

Draw the graph of $f(x) = \log x$.

Solution

The graph of $f(x) = \log x$; $x > 0$.

The graph of $f(x)$ will be constructed by the same method we used to draw the graphs of some other logarithmic functions.

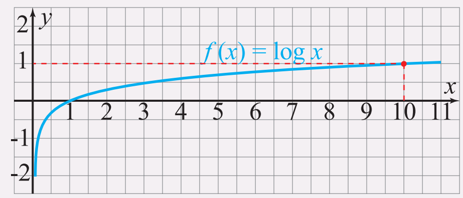


Figure 20.

We have seen that $\log_{10}1 = 0$ and $\log_{10}10 = 1$ if $1 < x < 10$, then $0 < \log x < 1$.

(In base 10 logarithm the logarithm of numbers which are between 1 and 10 is between 0 and 1)

Since common logarithms are mainly used for computations we will have a table of values of $\log^x$ for some x such that $1 \leq x \leq 10$. Moreover, using the standard notation or scientific notation we can compute $\log x$ for all $x > 0$.

Example 38

Use the logarithm table at the end of this book or a calculator to find the logarithms of the following numbers.

(a) log 8.73 (b) log 5.68 (c) log 2.51

Solution

(a) $\log 8.73 = 0.9410$

(b) $\log 5.68 = 0.7543$

(c) $\log 2.51 = 0.3997$

Example 39

Use standard notation and logarithm table to find the following logarithms in base 10.

(a) log 258 (b) log 0.0259

Solution

(a) $\log 258 = \log 2.58 \times 10^2$
$= \log 2.58 + \log 10^2$
$= 0.4116 + 2.$

Note: Here the decimal part of the logarithm (0.4116) is called **mantissa** and the integral part (2) is called **characteristics**

(b) $\log 0.0259 = \log 2.59 \times 10^{-2}$
$= \log 2.59 + \log 10^{-2}$
$= 0.4133 + (-2).$

Note: 0.4133 + −2 can also be written as –1.58678.

The first answer 0.4133 + (−2) is especially important way of writing because it is important for computations by using logarithmic tables.

EXAMPLE 40

If log 2 = 0.3010 and log 3 = 0.4771, then find the following logarithms **without using logarithm table or a calculator.**

(a) $\log^6$ (b) log 125 (c) log 0.00036

Solution

(a) $\log 6 = \log(2 \times 3) = \log 2 + \log 3$

$= 0.3010 + 0.4771$

$= 0.7781.$

(b) $\log 125 = \log \frac{125}{1000} \times 1000$

$= \log \frac{1}{8} + (3) = -3\log 2 + 3$

$= -3 \times 0.3010 + 3$

$= -0.9030 + (3) = 0.0970 + 2.$

(c) $\log 0.00036 = \log 6^2 \times 10^{-5}$

$= 2\log 6 + \log 10^{-5}$

$= 1.5562 + (-5)$

$= 0.5562 + (-4).$

I. Anti-Logarithms

EXAMPLE 41

If log x = 0.5955, then find the value of x.

Solution

Here x is called anti-logarithm of 0.5955. If you refer this in anti-logarithm table or in the absence of this table if you refer in logarithm table you can see log 3.94 = 0.5955 and 3.94 is anti-logarithm of 0.5955.

EXAMPLE 42

If log x = 3.7372, then what is x?

Solution

$\log x = \log 5.46 \times 10^3$.............. (note log 5.46 = 0.7372)

Therefore, $x = 5.46 \times 10^3 = 5460.$

EXAMPLE 43

If $\log x = -4.6973$, then find the value of x.

Solution

First let's get a positive mantissa

$$\begin{array}{r} 5.0000 + (-5) \\ -4.6973 \\ \hline 0.30271 + (-5) \end{array} \text{ (adding 5 and subtracting 5)}$$

Therefore $\log x = \log 2.01 \times 10^{-5}$ .... (note 0.3027 is closer to log 2.01)

$x = 2.01 \times 10^{-5} = 0.0000201$.

II. Natural Logarithm

1. If the base of the logarithm is **e,** then it is called natural logarithm where e is an irrational number; **e** = 2.71828182845904 . . .
The natural logarithm of a number x is denoted by $\ln x$.

EXAMPLE 44

(a) $\ln (86) = 4.454347$

(b) $\ln (2.7182818) = 1$

(c) $\ln e^3 = 3$

(d) $e^{\ln 4} = 4$

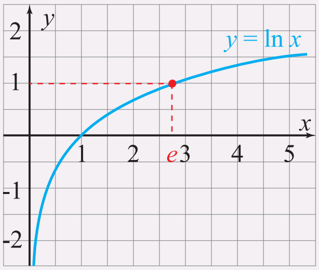


Figure 21.

EXERCISES

1. Write the following numbers in standard notation
   (a) 529 (b) 23768 (c) 0.000004721
2. Write the following numbers in standard notation to three significant digits
   (a) 0.0004792 (b) 479836 (c) 0.8926
3. Use base 10 logarithm table to find
   (a) log 2752 (b) log 0.0007259 (c) log 19.2931
4. Between what two consecutive integers do we find x if
   (a) $10^x = 0.004712$ (b) $10^x = 97.521$ (c) $10^x = 315000$

5. If log 2 = 0.3010 and log 3 = 0.4771, then find the following without using logarithm tables

   (a) log 0.06
   (b) log 12000
   (c) $\log\left(\frac{1}{36}\right)$
   (d) log 6.4
   (e) log 0.0064

6. Compute the following by using Base –10 logarithm table

   (a) 253×4561000
   (b) $328630 \div 981$
   (c) $\left(\sqrt[3]{256}\right)^5$
   (d) $\frac{2751 \times 836}{0.921}$

Application problems on logarithmic functions

Example 45

A man puts L$ 1000 in saving account at 8% interest compounded annually. How long will take for his money:

(a) to double the money?

(a) to be L$ 4660.95?

Solution

(a) In the formula $A = P\left(1+\frac{r}{100}\right)^n$,

$A = 2000$, $P = 1000$, $r = 8$ and n unknown.

$\Rightarrow 2000 = 1000\left(1+\frac{8}{100}\right)^n$

$\Rightarrow 2 = 1.08^n$, dividing both sides by 1000.

$\Rightarrow n = \log_{1.08} 2$

$= \frac{\log 2}{\log 1.08} = \frac{0.3010}{0.0334}$

$= 9.0119$

$\Rightarrow n \cong 9$, it needs 9 years to double the money.

(b) $A = 4660.95$, $P = 1000$, $r = 8$ and n required to be calculated.

$\Rightarrow 4661 = 1000\ (1.08)^n$

$\Rightarrow \frac{4661}{1000} = (1.08)^n$

$\Rightarrow 4.661 = 1.08^n$

$\Rightarrow \log_{1.08} 4.661 = n$

$\Rightarrow n = \dfrac{\log\ 4.661}{\log 1.08}$

$\Rightarrow n = \dfrac{0.6685}{0.0334}$

$= 20.0149$

$\Rightarrow n \cong 20$

$\Rightarrow$ It needs 20 years to reach L$ 4661.

EXAMPLE 46

A money is put in a saving account with 9% interest compounded annually. Calculate the year in which the money will double itself.

Solution

$A = 2P$ and $r = 9$

$\Rightarrow 2P = P\left(1 + \dfrac{9}{100}\right)^n$

$\Rightarrow \dfrac{2P}{P} = (1.09)^n$

$\Rightarrow 2 = 1.09^n$

$\Rightarrow n = \log_{1.09} 2$

$\Rightarrow n = \dfrac{\log\ 2}{\log 1.09} = \dfrac{0.3010}{0.0374}$

$= 8.049$

$\therefore$ It needs 8 years to double the money.

EXAMPLE 47

L$ 6,400 is invested at 8% annual interest compounded quarterly. How many years are required for his money to be L$ 10,294?

Solution

Substitute $A = 10{,}294$, $P = 6{,}400$, $r = 8\% = 0.08$ and $n = 4$ in the formula

$A = P\left(1\ +\ \dfrac{r}{n}\right)^{rt}$ to find t.

$10{,}294 = 6{,}400\ \left(1 + \dfrac{0.08}{4}\right)^{4t}$

$\Rightarrow \dfrac{10{,}294}{6{,}400} = (1.02)^{4t}$

$$\ln\left(\frac{10{,}294}{6{,}400}\right) = 4t\ \ln(1.02)$$

$\Rightarrow 0.4752 = 4t\,(0.0198)$

$\Rightarrow 4t = 24$

$\Rightarrow t = 6$ years.

Example 48

What should be the rate of interest for an amount L$ 7,450 to become double in 11 years and 6 months if the amount is compounded continuously?

Solution

It is given that $A = 2P$ and $t = 11.5$

Thus, $A = Pe^{rt}$

$\Rightarrow 2P = Pe^{11.5r}$

$\Rightarrow 2 = e^{11.5r}$

$\Rightarrow \ln 2 = 11.5r$

$$\Rightarrow\ r = \frac{\ln 2}{11.5r}$$

$$\Rightarrow\ r = \frac{0.6931}{12}$$

$= 0.06$

$\Rightarrow r = 6\%$.

III. Population Growth

Example 49

The population of a given country is 26,000,000. If the population increases at the rate of 2.5% every year, find the population after 15 years.

Solution

In this problem we use the formula

$y = P\left(1+\frac{r}{100}\right)^n$, where P is the population in 2000

$r = 2.5$, $n = 15$ and y the population in 2015.

$$\Rightarrow y = 26000000\left(1+\frac{2.5}{100}\right)^{15}$$

$$\Rightarrow \log y = \log\left(26000000\left(1+\frac{2.5}{100}\right)^{15}\right)$$

$$= \log 2.6 \times 10^7 + \log (1.025)^{15}$$

$$= 7 + \log 2.6 + 15\log 1.025$$

$$= 7 + 0.4150 + 15\ (0.0107)$$

$$= 7.5755$$

$$\Rightarrow y = 10^7 \times \text{anti-log } 0.5755$$

$$= 10^7 \times 3.762703523$$

$$= 37627035.23.$$

Therefore, the population of the country is expected to be 37627035.23 after 15 years.

IV. Depreciation of Value

$$V_t = V_0\left(1-\frac{r}{100}\right)^t, \text{ where}$$

V_0 = the value at a certain time

r = the rate of depreciation per year

V_t = the value at the end of t-years

Example 50

A computer purchased for L$ 11000 depreciates at the rate of 15% per annum. Find the depreciated value after 8-years.

Solution

$V_0 = 11000$, $r = 15$, $t = 8$

Required: V_8

$$V_t = V_0\left(1-\frac{r}{100}\right)^t$$

$$V_8 = 11000\left(1-\frac{15}{100}\right)^8$$

$$= 11000\ (1 - 0.15)^8 = 11000\ (0.85)^8$$

$$\Rightarrow \log V_8 = \log 11000\ (0.85)^8$$

$$= \log 11000 + \log(0.85)^8$$

$$= \log 1.1 \times 10^4 + 8 \log (0.85)$$

$= 4 + 0.041 + 8 (\log 8.5 \times 10^{-1})$

$= 4.0414 + 8 (\log 8.5 + (-1))$

$= 4.0414 + 8 (0.9294 + (-1))$

$= 3.4766.$

$\Rightarrow V_8 = \text{antilog } (3.4766)$

$= 2996.40.$

Example 51

A bacterial culture which is growing exponentially increases from 4 g to 6 g in 12 hours. How long does it take the culture to double its mass?

Solution

$y = y_0 e^{kt}$

At $t = 0, y_0 = 4$ g $\quad \Rightarrow y = 4e^{kt}$

At $t = 12, y = 6$ g $\quad \Rightarrow 6 = 4e^{k(12)}$

$$\Rightarrow \frac{3}{2} = e^{12k} \Rightarrow e^k = \left(\frac{3}{2}\right)^{\frac{1}{12}}$$

$$y = 4 \times \left(\frac{3}{2}\right)^{\frac{t}{12}}.$$

To double its mass, $y = 2y_o = 8 \Rightarrow 4 \times \left(\frac{3}{2}\right)^{\frac{t}{12}} \Rightarrow 2 = \left(\frac{3}{2}\right)^{\frac{t}{12}}$

$$\Rightarrow \ln 2 = \frac{t}{12} \ln\left(\frac{3}{2}\right) \Rightarrow t = 12 \frac{\ln 2}{\ln\left(\frac{3}{2}\right)}.$$

Reading from a table of logarithms gives the approximate value of t

$\Rightarrow t \simeq$ 20 hours and 30 minutes.

Example 52

A radioactive isotope decreases from 6 mg to 5 mg in 20 years. Find the half-life of the isotope.

Solution

$t = 0, y_0 = 6 \Rightarrow y = 6e^{-kt}$

$t = 20$ years $\Rightarrow y = 5$

$$\Rightarrow e^{-k} = \left(\frac{5}{6}\right)^{\frac{1}{20}} \Rightarrow y = 6 \times \left(\frac{5}{6}\right)^{\frac{t}{20}}$$

$$y = \frac{1}{2}y_o \Rightarrow \frac{1}{2}(6) = 6 \times \left(\frac{5}{6}\right)^{\frac{t}{20}}$$

$$\Rightarrow \frac{1}{2} = \left(\frac{5}{6}\right)^{\overline{20}} \Rightarrow \frac{1}{2} = \left(\frac{5}{6}\right)^{\frac{t}{20}}$$

$$\Rightarrow \frac{t}{20} = \frac{\ln\frac{1}{2}}{\ln\left(\frac{5}{6}\right)}$$

$$\Rightarrow t = 20\frac{\ln\frac{1}{2}}{\ln\left(\frac{5}{6}\right)}$$

$\Rightarrow t = 76$ years.

Exercises

1. Solve $2^x = 5^{x+1}$
2. Solve $2^x.\ 3^x = 12^x$
3. Solve $4^{2x+1} = 3^{x-2}$
4. Solve $7^{-3x+1} = 2^{2x+3}$
5. A man invested L$ 5000 in a business that pays 5% interest compounded annually. Find the amount at the end of
   (a) the first year
   (b) the 10^{th} year
6. A man invested money in a bank that pays 8% interest compounded semiannually. Calculate the time on which the money will have an interest equal to
   (a) half the money
   (c) equal to the money
   (b) $\frac{3}{4}$ of the money
7. The population in colony A was twice the population in colony B. If in 15 periods the populations in the two colonies will be equal and if the growth rate in colony A is 1.5%, find the growth rate in colony B.
8. Two countries of the same population size are having 1.8% and 2.7% growth rate. When will the population of one will double the other?
9. Suppose radioactive isotope decays exponentially with time. Initially there were 6 mg and after 20 years 0.8 mg decayed.
   (a) What is its half - life?

(b) How much will remain after 100 years?

10. A money is put in a saving account 9% compounded annually. Calculate the year in which the money will double itself.

11. The size of bacteria in a culture doubles in 10.4 minutes. In how many minutes will the population triple?

12. A man invests in a saving account with annual interest rate of 6.8% compounded monthly. After 5 years the balance reaches L$ 10,527. Find the initial investment.

Key Terms

- Base
- Compound interest
- Continuous compounding
- Common logarithm
- Characteristic
- Exponential functions
- Exponential equations
- Exponents
- Growth and decay
- Half - life
- Logarithmic functions
- Logarithmic equations
- Mantissa
- Natural logarithm
- Power
- Population growth

Summary

- Exponential function
  $f(x) = a^x$; $a > 0$ $a \neq 1$ is said to be exponential function
  $f(x) = e^x$ is said to be the natural exponential function.
- Properties of the exponential functions $f(x) = a^x$
  - The graph passes through the point at (0, 1)
  - The domain is the set of all real numbers.
  - The range is the interval $(0, \infty)$

  (i) When $a > 1$, The graph is smooth and continuous
    - The x-axis is the horizontal asymptote
    - It is an increasing function
- Compound Interest formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- Continuous compounding formula: $A = Pe^{rt}$

Exponential Growth and Decay Formula

$y = y_o e^{kt}$

- Logarithmic function

  $f(x) = \log_a x$ where $x > 0$, $a > 0$ and $a \neq 1$ is said to be logarithmic function.

  $y = \log_a x$ if and only if $x = a^y$

  $f(x) = \ln x$ is said to be the natural logarithmic function.

  Exponential and logarithmic functions are inverses of each other.

- Properties of the logarithmic function $f(x) = \log_a x$
  - The domain is the interval $(0, \infty)$
  - The range is the set of all real numbers.
  - The range is the set of all real numbers.
  - The graph passes through the point at $(1, 0)$
  - The graph is smooth and continuous.

  (i) When $a > 1$
  - The y-axis is the vertical asymptote.
  - It is an increasing function.

  (ii) When $0 < a < 1$
  - the positive y-axis the vertical asymptote.
  - it is a decreasing function.

Exercises

I. True or False Items

1. $(-a)^n = a^n$ if $n \in \mathbb{Z}^+$, $a \in \mathbb{R}$, $a \neq 0$
2. The range of $y = 2^{x-1}$ is $\{y : y \geq 1\}$
3. The asymptote to the graph of $y = 4 \times 3^x$ is $y = 0$.
4. $\log_{\frac{1}{2}} x > \log_{\frac{1}{3}} x$ if $0 < x < 1$
5. If $1 < x < 10$, then $0 < \log x < 1$
6. $y = \log_{\frac{1}{2}} x$ is inverse of $y = 2^{-x}$

II. Work out

7. If $2\log_{\frac{1}{3}} a = x$ and $\log_9 b = y$, then $\log_3 ab$ is equal to

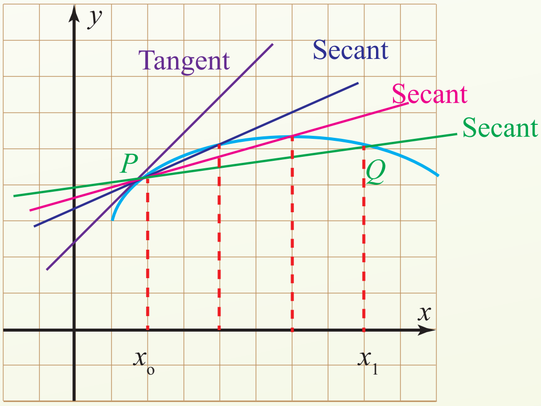
8. If $f(x)=\left(\frac{1}{3}\right)^x$, then
   (a) The y-intercept is ______
   (b) $f(x)>1$ if x is __________
   (c) Range of f is ___________
9. If $\log_a 4 = x$, $\log_a 3 = y$ and $\log_a b = z$, then $\log_b 36$ is equal to _____
10. If $y = 10^{\log_{10} 7}$, then $y =$______________
11. The Solution of $\log_{\sqrt{3}} N = x+1$ and $\log_{27} N = \frac{x^2}{3}$ for a natural number N is:
12. If $a > 1$, then for what values of x is $\log_a x < 0$?
13. If log 2.5 = 0.3979 and log 4.9 = 0.6902, then what is $\log\left(\frac{625\times 7^3}{\sqrt{4900}}\right)$
14. Solve $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$
15. Solve $13^{3x+2} = \left(\frac{1}{169}\right)$
16. Solve each of the following
   (a) $2^x = 128$
   (b) $\left(\frac{5}{2}\right)^{x+2}\left(\frac{2}{5}\right)^{2x-5} = (0.4)^{3x+5}$
17. Draw the graph of each of the following functions and their inverses. Determine the domain, the range and the asymptotes.
   Find the x or y-intercept. (Questions 41–45)
   (a) $f(x) = 6^x$
   (b) $f(x) = 0.4^x$
   (c) $f(x) = -3 + 2^{1-3x}$
   (d) $f(x) = 5 - 3^{2x+1}$
18. Solve $\log_{0.3}\left(\frac{27}{1000}\right) = \left(\frac{1}{81}\right)^x$
19. If $\log_{27} a = x$ and $\log_{\left(\frac{1}{9}\right)} b = y$, find $\log_3 ab$
20. Solve $\left(\frac{4}{9}\right)^x \cdot \left(\frac{27}{8}\right)^{x-1} = \frac{\log_{10} 4}{\log_{10} 8}$
21. The interest found when L\$ 500 was invested after 6 years deposit was L\$ 170. Find the rate of interest.
22. In a country the population before 30-years was 2.6×10^6. At present the population is 5.4536×10^6. Find the rate at which the population increases every year.

M12CH22

CHAPTER 22

DIFFERENTIATION AND INTEGRATION

Chapter Contents



Chapter Outcomes

Upon the completion of this chapter learners will able to:

- define , discuss and apply the concept of difference quotient;
- review slopes, tangent lines and derivatives;
- define and discuss the concept of limits;
- define and apply the concept of differentiation;
- define and discuss the concept of integration;
- find area under a curve;
- find indefinite integrals of simple polynomial and trigonometric functions.

DEFINITION

A coordinate plane is a plane whose points have been placed in one- to-one correspondence with the set of ordered pairs of real numbers, $\mathbb{R} \times \mathbb{R}$.

Distance in a Coordiante Plane

Theorem: (Distance formula)

The distance $|P_1P_2|$ between the points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

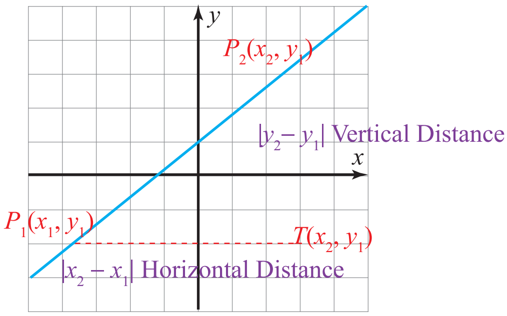


Figure 1.

Proof: ΔP_1TP_2 is right with $m(\angle T) = 90^o$

$$(P_1T)^2 + (P_2T)^2 = (P_1P_2)^2$$

$$|x_2 - x_1|^2 + |y_2 - y_1|^2 = (P_1P_2)^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(*Distance formula between* P_1 *and* P_2)

EXAMPLE 1

Find the length of a line segment whose end points have coordinates (3, 1) and (–1, 4).

Solution

$$d = \sqrt{(3-(-1))^2 + (1-4)^2}$$

$$= \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

EXAMPLE 2

How far is the origin from the point (–3, –1)?

Solution

$$\sqrt{(0-(-3))^2 + (0-(-1))^2} = \sqrt{3^2 + 1^2}$$

$$\sqrt{10}.$$

EXAMPLE 3

Find the value of k if the distance of $(1, k)$ from $(–2, 3)$ is $2\sqrt{5}$.

Solution

$$\sqrt{(1-(-2))^2+(k-3)^2}=2\sqrt{5}$$

$3^2 + (k-3)^2 = (2\sqrt{5})^2$

$9 + k^2 - 6k + 9 = 20$

$k^2 - 6k - 2 = 0$

$$k=\frac{6\pm\sqrt{36+8}}{2}$$

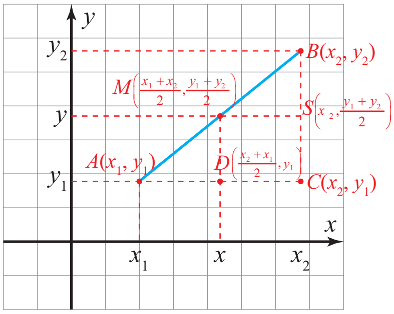
$$=\frac{6\pm 2\sqrt{11}}{2}=3\pm\sqrt{11}.$$

Mid–Point Formula

Theorem: (Mid–point formula)

The mid-point $M(x, y)$ of the line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates.

$$x=\frac{x_1+x_2}{2} \text{ and } y=\frac{y_1+y_2}{2}.$$



Proof:

Let D be midpoint of $\overline{AC}$. Then D has coordinate $\left(\frac{x_1+x_2}{2}, y_1\right)$ and Let S be midpoint of $\overline{BC}$. Then S has coordinate $\left(x_2, \frac{y_1+y_2}{2}\right)$.

But $\overline{MD} // \overline{BC}$ hence D and M have the same abscissa and $\overline{MS} // \overline{AC}$ hence M and S have the same ordinate,

$$\therefore x=\frac{x_1+x_1}{2} \text{ and } y=\frac{y_1+y_2}{2}.$$

Example 4

Find the coordinates of the mid points of the line segment whose end points have coordinates (–4, 1) and (3, –7),

Solution

$$\frac{x_1+x_2}{2}=\frac{-4+3}{2} \text{ and } \frac{y_1+y_2}{2}=\frac{1+-7}{2}=-0.5 \text{ and } -3.$$

Therefore, the mid point has coordinate (–0.5, –3).

Example 5

One end point of a line segment has coordinate (–1.5, 4) and its midpoint (–2, –1). Find the coordinate of the second end point.

Solution

Take $(x_1, y_1) = (-1.5, 4)$ and $(x, y) = (-2, -1)$

$$-2 = \frac{-1.5 + x_2}{2} \text{ and } -1 = \frac{4 + y_2}{2}$$

$x_2 = -2.5$ and $y_2 = -6$.

Therefore, the second end point has coordinate (–2.5, –6).

Trisecting a Line Segment

Suppose A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively. If M and N are between A and B such that $AM = MN = NB$, then M and N have coordinates $\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right)$ and $\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right)$ respectively.

Dividing a line segment in the ratio m:n

If $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ are end points of line segment P_1P_2 and R a point between P_1 and P_2 that divides $\overline{P_1P_2}$ in the ratio m:n, then the coordinates of R are:

$$\left(\frac{nx_1 + mx_2}{m + n}, \frac{\text{ny}_1 + my_2}{m + n}\right).$$

Example 6

If P (4, 7) and Q (39, 56) are end points of $\overline{PQ}$ what is the coordinate of point R on $\overline{PQ}$, that divides PQ in the ratio 5:2?

In the above formula $\left(\frac{nx_1 + mx_2}{n + m}, \frac{ny_1 + my_2}{n + m}\right)$, substitute $(x_1, y_1) = (4,7)$ and $(x_2, y_2) = (39,56)$, $m = 5$ and $n = 2$.

$$\therefore R = (x_o, y_o) = \left(\frac{2 \times 4 + 5x39}{2 + 5}\right), \left(\frac{2 \times 7 + 5 \times 56}{2 + 5}\right) = (29, 42).$$

Exercises

1. Find distance between P_1 and P_2 if P_1 and P_2 have coordinates
   (a) (–6, 0), (–7, –3)
   (b) (–7, –3) (–4, 9)
   (c) Origin and (a, b)
   (d) (–3, –2), (–11, –1)

2. Find mid point coordinates of $\overline{P_1P_2}$ if P_1 and P_2 are as given in question 1 above.
3. Find the points of trisection of the line segment with end points (–5, –2) and (–35, – 23).
4. The line segment with end points A and B having coordinates (3, –1) and (4, 2) respectively is extended to C. Find the coordinates of C for each of the following conditions. If B is between A and C

   (a) $AC = 2AB$ (b) $AC = 10AB$ (c) $AC = kAB, k > 1$
5. The vertices of a triangle have coordinates $A(-1, 4)$, $B(3, 7)$ and $C(4, -1)$. Find the lengths of its medians. Find the coordinates of the centroid.
6. Show that triangle with vertices A (3, 2), B (0, 0) and C (6, 0) is isosceles triangle.
7. Show that the points A (a, b), B (p, q) and $C(2a - p, 2b - q)$ can't determine the vertices of a triangle.
8. If R is appoint on $\overline{PQ}$ with coordinates of $P(-5, 11)$ and the coordinate of $Q(31, 47)$ find the coordinate of R if:

   (a) R is a point that divides $\overline{PQ}$ in the ratio 2:4.

   (b) R is a point that divides $\overline{PQ}$ in the ratio 1:11.

   (c) R is a point that divides $\overline{PQ}$ in the ratio 4:5.

The slope m of a non-vertical line which passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ where $x_1 \neq x_2$ is defined as the ratio of vertical increase (increase in y) to horizontal increase (increase in x).

Slope,

$$m = \frac{\text{Vertical increase}}{\text{horizontal increase}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

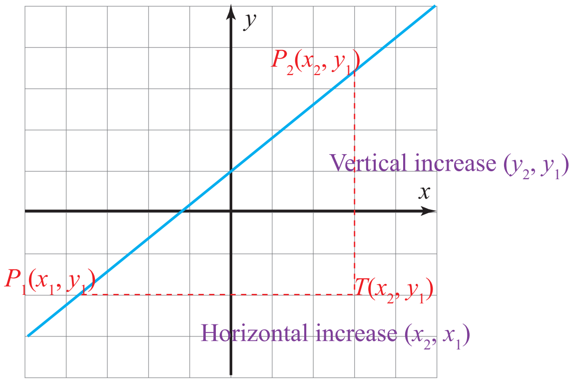


Figure 2.

Note: It doesn't matter which is represented by (x_1, y_1) and which by (x_2, y_2) since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}$.

EXAMPLE 7

Find the slope of the line which passes through the points

(a) $A(6, -4)$ and $B(3, 3.5)$.

(b) $H(-15, 3)$ and $K(2, 3)$,

(c) $N(2, 1.5)$ and $M(2, -12)$

Solution

(a) Take $(x_1, y_1) = (6, -4)$ and $(x_2, y_2) = (3, 3.5)$

$$\therefore\ m = \frac{3.5-(-4)}{3-6} = \frac{7.5}{-3} = -2.5.$$

(b) The Slope of HK, $m = \frac{3-3}{2-(-15)} = 0$.

(c) $m = (-12-1.5)/(2-2) = 10.5/0$, which is undefined.

Note: Two points of the form $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$ determine a vertical line (Note that they have the same abscissas)

- Slope of a vertical line is not defined.
- $\frac{y_2 - y_1}{x_1 - x_1}$ is undefined.
- Two points of the form $P_1(x_1, y_1)$ and $P_2(x_2, y_1)$ determine a horizontal line. (Note that they have the same ordinates)
- Slope of a horizontal line is zero, $\frac{y_1 - y_1}{x_2 - x_1} = 0$.

EXAMPLE 8

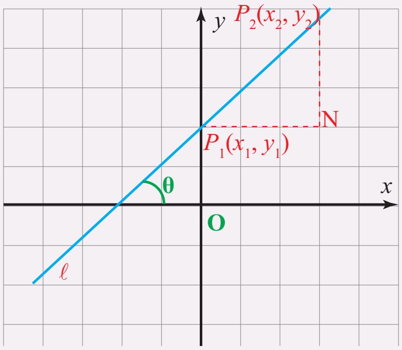


Figure 3.

Find the slope of a non-vertical line ℓ which is inclined to the positive direction of the x–axis.

Slope of ℓ, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{P_2N}{P_1N} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$.

$\therefore m = \tan\theta$

(θ ***is called the angle of inclination of line*** ℓ)

EXAMPLE 9

Write the slope of the line which is inclined at the following angles to the positive direction of the x-axis

(a) 30^{o} (b) 45^{o} (c) 90^{o}

Solution

(a) $\tan 30^{o} = \frac{\sqrt{3}}{3}$ which is the slope

(b) $\tan 45^{o} = 1$ which is the slope.

(c) $\tan 90^{o}$ . . . . . undefined (no slope)

Collinear Points (Points on the same straight line).

Three points are collinear if the slope of the line which passes through the first point and the second point is equal to either:

(i) the slope of the line which passes through the first point and the third point or

(ii) the slope of the line which passes through the second point and the third point.

EXAMPLE 10

Determine, by comparing their slopes, whether the three points with the given coordinates are collinear

(a) (–5, –2), (4, 7), (3, 6)

(b) $\left(\frac{1}{2}, 0\right)$, (2, 1), (8, –5)

(c) $\left(\frac{1}{3}, 0\right)$, (1, 2), (–1, –3)

Solution

(a) The slope m_1, between (–5, –2) and (4, 7) is

$$m_1 = \frac{-2-7}{-5-4} = \frac{-9}{-9} = 1.$$

The slope m_2 between (4, 7) and (3, 6) is

$$m_2 = \frac{7-6}{4-3} = 1.$$

Therefore, (–5, –2), (4, 7) and (3, 6) are collinear

(b) m_2 the slope between $\left(\frac{1}{2}, 0\right)$ and (2, 1)

$$m_2 = \frac{1-0}{2-\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

$$m_3 = \frac{-5-0}{8-\frac{1}{2}} = \frac{-5}{\frac{15}{2}} = \frac{-2}{3}$$ the slope between $\left(\frac{1}{2}, 0\right)$ and (8, –5).

Therefore, $\left(\frac{1}{2}, 0\right)$, (2, 1) and (8, –5) are not collinear.

(c) $m_1 = \dfrac{2-0}{1-\frac{1}{3}} = \dfrac{2}{\frac{2}{3}} = 3.$

$m_2 = \dfrac{-3-2}{-1-1} = \dfrac{-5}{-2} = \dfrac{5}{2}$; $m_1 \neq m_2$.

Therefore, $\left(\frac{1}{3}, 0\right)$, (1, 2) and (–1, –3) are not collinear points.

Exercises

1. What is the slope of a horizontal line
2. If the line through $P(a, b)$ and $Q(c, s)$ is vertical
   (a) What do you say about a and c?
   (b) What is the slope of the line
3. Find the slope of a line passing through the points
   (a) (5, 0) and (2, 7)
   (b) (–7, 3) and (3, –7)
   (c) (a, b) and (b, a); $a \neq b$
   (d) (–2, –7) and (–3, –2)
   (e) (0, 0) and $\left(\frac{1}{2}, \frac{2}{3}\right)$
   (f) $(-p, -q)$, $(-q, -2p)$
4. Determine whether each of the following points are collinear or not
   (a) (0, –5), (1, –2), (–3, –14)
   (b) (2, 3.75), (0.125, 8), (–4, –2.5)
   (c) $\left(\frac{-1}{2}, 5\right)$, $\left(3, \frac{5}{4}\right)$, $(11, -1)$
5. Find the value of a so that the line passing through $(2a + 1, a)$ and $(4, a^2 - 20)$ will have:
   (a) no slope
   (b) slope zero
   (c) a positive slope
   (d) a negative slope

Equations of A Line

Two Point Form Equation of a Line

Let ℓ be a line passing through two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, where $x_1 \neq x_2$, and let $Q(x, y)$ be any variable point on ℓ.

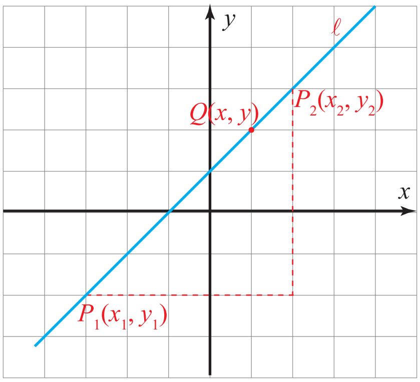


Figure 4.

You can see that slope of ℓ is independent of the points we choose on ℓ so the equation of the line can be defined as:

$$m = \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

So any point on ℓ satisfies the above equation.

EXAMPLE 11

Find the equation of the line passing through $P(1, 4)$ and $R(3, 10)$.

Solution

Let (x, y) be any point on PR

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow \frac{y-4}{x-1} = \frac{(10-4)}{(3-1)}$$

$$\frac{y-4}{x-1} = 3 \text{ or } y - 4 = 3x - 3 \qquad y = 3x + 1$$

$y = 3x + 1$ is the equation of the line through $(1, 4)$ and $(3, 10)$.

Slope Intercept form Equation of a Line

DEFINITION

If a line meets the x-axis at $(a, 0)$ and the y-axis at $(0, b)$, then $\boldsymbol{a}$ is called the x–intercept and $\boldsymbol{b}$ is called the y–intercept of the line.

Theorem:

If a line has slope m and a y-intercept b then the slope–intercept form of its equation is $\boldsymbol{y = mx + b}$.

Proof: $\frac{y - y_1}{x - x_1} = m;$

$y - y_1 = m(x - x_1)$

But the y–intercept is b.

Hence, the line passes through $(0, b)$.

$y - b = m(x - 0)$

Therefore, $y = mx + b$.

Example 12

Find the equation of a line with slope is 3 and y–intercept is –4.

Solution

$m = 3$ and $b = -4$ using

$y = mx + b$ we have

$y = 3x - 4$.

Example 13

Find the equation of a line whose slope is $\frac{-b}{a}$ and whose y-intercept is b.

Solution

$m = \frac{-b}{a}$ and $b = b$

Using $y = mx + b$ we have

$y = \frac{-b}{a}x + b \Rightarrow ay + bx = ba$ or $\frac{y}{b} + \frac{x}{a} = 1$ . . . . . Dividing both sides by ab.

Example 14

Find the equation of the line passing through (5, 2) and (–3, 1).

Solution

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow \frac{y-2}{x-5} = \frac{1-2}{-3-5} = \frac{1}{8} \ldots\ldots\ldots \left(\text{slope} = \frac{1}{8}\right)$$

$y = \frac{1}{8}x + \frac{11}{8}$ or $8y = x + 11$.

or you can follow the following

$$y = mx + b \begin{cases} x = 5 \text{ and } y = 2 \\ x = -3 \text{ and } y = 1 \end{cases}$$

$$\left.\begin{aligned} 2 &= 5m + b \\ 1 &= -3m + b \end{aligned}\right\}$$

Solving this simultaneous equation . . . .$8m = 1$ or $m = \frac{1}{8}$

$b = 2 - \frac{5}{8} = \frac{11}{8}$.

$\therefore\ y = \frac{1}{8}x + \frac{11}{8}$ or simply use $y = mx + b$ where $m = \frac{1}{8}$

$y = \frac{1}{8}x + b$

$2 = \frac{1}{8} \times 5 + b$. . . . .(substituting one of the points (5, 2))

$b = \frac{11}{8}$

Therefore, $y = \frac{1}{8}x + \frac{11}{8}$.

Example 15

Find the slope of the line $3x + 5y = 10$

Solution

Solving for y, we have $y = -\frac{3}{5}x + 2$, which compares with $y = mx + b$ with $m = -\frac{3}{5}$ and $b = 2$. Hence the slope is $-\frac{3}{5}$ and its y–intercept is 2.

Example 16

Find the slope and y-intercept of the line $Ax + By + C = 0$, $A, B \neq 0$.

Solution

Solving for y, we have $y = -\frac{A}{B}x - \frac{C}{B}$, which compares with $y = mx + b$ with $m = -\frac{A}{B}$ and $b = \frac{-C}{B}$.

Therefore, the slope is $-\frac{A}{B}$ and the y–intercept is $-\frac{C}{B}$.

Example 17

Find the equations of the two lines through the origin which are inclined 45° to the line $2x + 3y = 6$.

Solution

Let ℓ_1 *and* ℓ_2 be the two lines through the origin with $\angle ACO = 45°$ and $\angle CBO = 45°$

Let $m(\angle BOA) = \gamma$ and $m(\angle COA) = \theta$. Then the equation of ℓ_1 is $y = (\tan \gamma)x$ and ℓ_2 is $y = \tan \theta$.

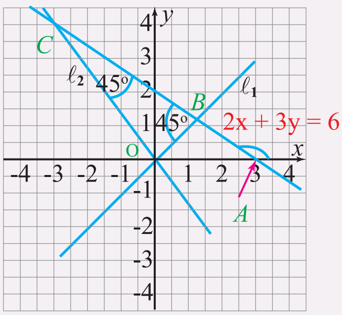


Figure 5.

Let α be the largest angle between $2x + 3y = 6$ and x–axis.

Then $\tan \alpha = -\frac{2}{3}$ = slope $\tan \gamma = \tan(180° - (135° + (180 - \alpha))$

$= \tan(180° - 135° - 180° + \alpha)$

$$= \tan(\alpha - 135°) = \frac{\tan\alpha - \tan 135°}{1 + \tan\alpha - \tan 135°} = \frac{-\frac{2}{3} - (-1)}{1 + -\frac{2}{3} - 1} = \frac{\frac{1}{3}}{\frac{5}{3}} = \frac{1}{5}$$

Therefore, ℓ_1 has equation $y = \frac{1}{5}x$ or $5y - x = 0$

$\tan\ \theta = \tan(180° - (45° + (180 - \alpha))$

$= \tan(180° - 45° - 180° + \alpha) = \tan(\alpha - 45°)$

$$= \frac{\tan\alpha - \tan 45°}{1 + \tan\alpha \tan 45°}$$

$$= \frac{-\frac{2}{3} - 1}{1 + -\frac{2}{3}.1} = -5$$

$\therefore \quad y = -5x$

$y + 5x = 0.$

General equation of a line

Equation of the form $Ax + By + C = 0$ where A and B are not both zero is called the general form of equation of a straight line.

Example 18

Find the general form of the equation of a line on the point (3, –5) and perpendicular to $7x - 3y = 2$.

Solution

Substituting

$x_0 = 3$, $y_0 = -5$, $A = 7$ and $B = -3$ in the equation

$A(y - y_0) - B(x - x_0) = 0$

$7(y - (-5)) - (-3)(x - 3) = 0$

$\therefore \quad 7(y + 5) + 3(x - 3) = 0$

$\therefore \quad 7y + 35 + 3x - 9 = 0$

$\therefore \quad 7y + 3x + 26 = 0.$

Parallel and Perpendicular Lines

A. Parallel Lines

Theorem:

Two lines are parallel if they have equal slopes.

Proof: Suppose ℓ_1 and ℓ_2 are parallel lines.

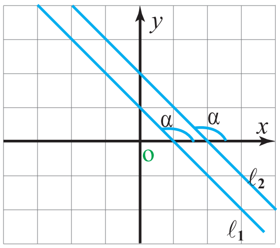


Figure 6.

If ℓ_1 and ℓ_2 are vertical lines, then their slopes doesn't exist, similarly if the slope of two lines doesn't exist, then they are vertical lines, hence they are parallel.

Suppose ℓ_1 and ℓ_2 are non–vertical parallel lines and let m_1 and m_2 be their respective slopes.

The angle formed by the x–axis and the lines ℓ_1 and ℓ_2 is equal, α measured from the x–axis counter clock wise to the lines

$\therefore$ the slope of $\ell_1 = m_1 = \tan\alpha = m_2 =$ the slope of ℓ_2

If $m_1 = m_2$

$m_1 = \tan\alpha$ and $m_2 = \tan\theta$

$\tan\alpha = \tan\theta,\ 0 \le \alpha, \theta \le \pi,\ \alpha, \theta \ne \dfrac{\pi}{2}$

$\therefore \alpha = \theta$

$\therefore \ell_1 \parallel \ell_2$.

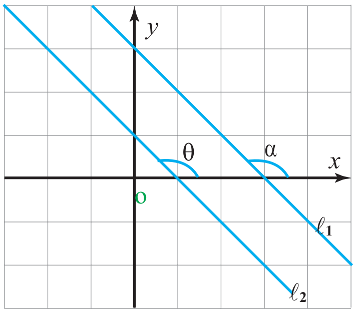


Figure 7.

EXAMPLE 19

Find the equation of the line passing through (3, 2) and parallel to the line whose slope is –2.

Solution

The slope of the required line is –2 since the line parallel to it has slope –2.

$\therefore y = mx + b$

$\therefore y = -2x + b$

$\therefore 2 = -2(3) + b$

$\therefore b = 8$

$\therefore y = -2x + 8$.

EXAMPLE 20

Find the equation of the line through (7, –3) which is parallel to the line whose equation is $y - 3x = 1$.

Solution

$y - 3x = 1$ has slope 3

$$\frac{y-(-3)}{x-7} = 3$$

$y = 3x - 24$ is the required line.

B. Perpendicular Lines

Two non-vertical lines are perpendicular if the product of their slopes is -1.

Proof: Suppose m_1 and m_2 are the slopes of ℓ_1 and ℓ_2 with $m_1 m_2 = -1$.

(i) If $\ell_1 \perp \ell_2$, then $m_1 m_2 = -1$

(ii) If $m_1 m_2 = -1$, then $\ell_1 \perp \ell_2$.

Example 21

Find the equation of the line passing through the origin and perpendicular to a line whose slope is 1.

Solution

Let m be the slope of the required line.

$m \times 1 = -1$ or $m = -1$

$\therefore y = mx$ and $m = -1$

$\therefore\ y = -x.$

Example 22

Find the equation of the line which is perpendicular to the line whose equation is $3x + 4y = 11$.

Solution

The slope of the line $3x + 4y = 11$ is $m = -\frac{3}{4}$

Let the slope of the required line be m'.

Then mm' $= -1$ or $-\frac{3}{4}m' = -1 \Rightarrow m' = \frac{4}{3}$

$\therefore y = m'x + b;\ m' = \frac{4}{3},\ y = -3,\ x = 1$

$\therefore b = -3 - \frac{4}{3} = -\frac{13}{3}$

$\therefore y = \frac{4}{3}x - \frac{13}{3} \Rightarrow 3y - 4x = -13.$

Theorem:

The equation of the line passing through (x_0, y_0) *and parallel to* $\ell : Ax + By + C = 0$ *is* $A(x - x_0) + B(y - y_0) = 0$

The equation of the line passing through (x_0, y_0) *and perpendicular to* $Ax + By + C = 0$ *is* $A(y - y_0) - B(x - x_0) = 0.$

The Distance of a Point from a Straight Line

Theorem:

The distance d of the point $A(x_0, y_0)$ *from the line* $\ell : Ax + By + C = 0$ *is given by the formula*

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

EXAMPLE 23

How for is the origin from the line $x + y - 1 = 0$

Solution

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}, \text{ where } A = 1, B = 1, C = 1 \text{ and } x_0 = 0 = y_0$$

$$\therefore d = \frac{|0 + 0 - 1|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}.$$

Distance between parallel lines

Consider the equations of two parallel lines given by $Ax + By + C = 0$ and $Ax + By + D = 0$ where A and B are not both zero.

The distance between two parallel lines is equal to the distance of a point on one of the lines from the other line.

Let $P = (x_0, y_0)$ be on the line $Ax + By + C = 0$.

Then, $Ax_0 + By_0 + C = 0$

$\Rightarrow Ax_0 + By_0 = -C$

The distance of P from $Ax + By + D = 0$ is $d = \dfrac{|Ax_0 + By_0 + D|}{\sqrt{A^2 + B^2}} = \dfrac{|D - C|}{A^2 + B^2}$.

EXAMPLE 24

Find the distance between the following pairs of lines.

(a) $8x + 6y + 9 = 10$ and $8x + 6y - 1 = 0$.

(b) $y = 3x - 5$ and $y = 3x + 7$.

(c) $y = \frac{3}{4}x$ and $y = \frac{3}{4}x = 5$.

Solution

Clearly the pairs of lines in questions a, b and c are parallels. See the slopes of the lines.

(a) $d = \frac{|9-(-1)|}{\sqrt{8^2+6^2}} = \frac{|10|}{10} = 1.$

(b) $y = 3x - 5$ and $y = 3x + 7$ can be written as $3x - y - 5 = 0$ and $3x - y + 7 = 0$.

(c) Thus $d = \frac{|-5-7|}{\sqrt{3^2+(-1)^2}} = \frac{12}{\sqrt{10}}$.

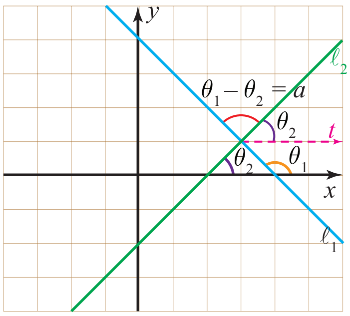
(d) $\frac{3}{4}x$ and $y = \frac{3}{4}x + 5 = 0 \Rightarrow 4y - 3x = 0$

$\Rightarrow d = \frac{|5-0|}{\sqrt{3^2+(-4)^2}} = 1.$

The angle between two straight lines

Consider lines ℓ_1 and ℓ_2 with slopes m_1 and m_2 respectively and the angels θ_1 and θ_2 between the x–axis and ℓ_1 and ℓ_2 measured from X counter clockwise to the lines.

Then $m_1 = \tan\theta_1$, and $m_2 = \tan\theta_2$, angle of inclination & slope. Let α be the angle between the lines.



Therefore, $\alpha = \theta_1 - \theta_2$

By constructing t parallel to the x – axis you can see the following relations.

you can see that

$\therefore\ \alpha = \theta_1 - \theta_2$

$\therefore \tan\alpha = \tan(\theta_1 - \theta_2)$.

$$= \frac{\tan\ \theta_1 - \tan\ \theta_2}{1 + \tan\ \theta_1\ \tan\ \theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

If one of the angles between the lines is α then the other angle is $(180^0 - \alpha)$ so that.

$\tan(180^0 - \alpha) = -\tan\alpha = -\frac{m_1 - m_2}{1 + m_1 m_2}$. Hence $\tan\alpha = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$.

EXAMPLE 25

Find the angle between the lines $3x - 5y + 2 = 0$ and $10x + 6y - 1 = 0$.

Solution

$\tan\ \alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ But $m_1 = \dfrac{3}{5}$ and $m_2 = -\dfrac{5}{3} \Rightarrow \tan\ \alpha = \dfrac{\frac{3}{5} - \left(-\frac{5}{3}\right)}{\left(\frac{-5}{3}\right)\left(\frac{3}{5}\right)} \Rightarrow \alpha = \dfrac{\left(\frac{34}{15}\right)}{0}$ which does not exist.

$\Rightarrow \alpha = 90^\circ$ since $0^\circ < \alpha < 180^\circ$.

Therefore, the angle between the line is 90° or that they are perpendicular.

EXAMPLE 26

Find the angle between the lines l_1 and l_2 with equations

$l_1 : y = 3x - 2$ and $l_2 : y = -2x + 5$

Solution

The slope of $l_1 = m_1 = 3$ and the slope of $l_2 = m_2 = -2$

Let θ be the angle between the lines

Then, $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{3-(-2)}{1+3\times -2} = \dfrac{5}{-5} = -1.$

$\theta = \tan^{-1}(-1) = 180^\circ - 45^\circ = 135^\circ$.

Therefore, the angle between the lines is 135° or 45°.

EXERCISES

1. What is the equation of the line with
   (a) slope –3 and y–intercept $\dfrac{1}{2}$
   (b) y–intercept 3 and x–intercept 5
   (c) slope 5 and passing through (2, 4)
2. What is the equation of the line passing through (–3, 4) and parallel to the line $3x + 5y - 2 = 0$?
3. How far is the point (3, –2) from the line $2x + 3y = 5$?
4. How far is the line $y = mx + c$ from the line $y = mx + b$?
5. What is the equation of the line passing through (–5, 2) and perpendicular to the line $2x - 5y + 2 = 0$?

6. If the distance from the point $P(k, 1)$ to the line $y = 3x + k$ is 3, find the value(s) of k.
7. Find the equation of the perpendicular bisector of the line segment joining the points $P(-1, 4)$ and $Q(3, 5)$.
8. Find the equation of the line passing through the point $P(-3, 2)$ that is parallel to the line passing through the points $Q(7, -2)$ and $R(1, 5)$.
9. Find the distance between each of the following pair of lines.
   (a) $y = 3x + 1$ and $y = 3x - 5$
   (b) $3x - 4y - 11$ and $4y - 3x + 7 = 0$
10. Find the measure of the acute angle between each of the following pairs of lines if it is not right angle.
    (a) $y = -1$ and $x = \sqrt{3}$
    (b) $y = x$ and $y = 1 - x$
    (c) $y = x + 5$ and $x = \sqrt{7}$
    (d) $y = 3x + 1$ and $3y + x = 1$
    (e) $y = 6 - x\sqrt{3}$
    (f) $y + 2x = 5$ and $x - y = 4$
    (g) $y = x + 3$ and $2y + \left(3 + \sqrt{3}\right)x + 5 = 4$
    (h) $2y - 5x = 6$ and $y + 2x + 1 = 0$
11. Suppose l_1 and l_2 are perpendicular lines intersecting at $(-3, 5)$. If the angle of inclination of l_1 is 45^o. What is the equation of l_2?
12. The acute angle between lines intersecting at $(3, 2)$ is 30^o. If the angle of inclination of one of the line is 150^o, find the equations of the lines.

Introduction

Calculus is a branch of mathematics which is applied in a wide variety of areas in mathematics and other fields such as engineering, science and commerce. In this section we shall introduce the two branches of calculus differential calcualus and integral calculus. In the first part, we shall give the defination of dervative based on the concept of limits and then find the derviatives of polynomial functions and simple trigonometric functions.

In the last section, we shall approximate the areas of a rgion under a curve, define the indefinite integral as a reverse process of differentiation and use the definite integral to calculate the area of a region under a curve.

The derivative

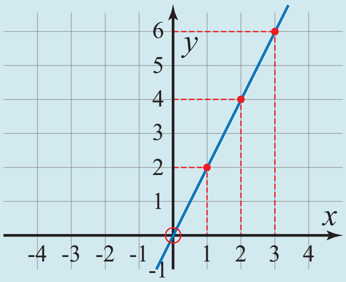
Difference Quotient

ACTIVITY 1

Consider the graph of $y = 2x$ on the interval (1, 3)

As x increase from $x = 1$ to $x = 3$, y is increasing from $y = 2$ to $y = 6$. Here, we see that an increase of 2 unit in x has shown an increase of 4 units in y. In this sense we say that the coverage rate of change of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{6-2}{3-1} = \frac{2}{4} = \frac{1}{2}.$$



1. Find the average rate of change of y with respect to x. When:
   (a) $y = 3x$ as x-increases from $x = 2$ to $x = 5$.
   (b) $y = x^2$ as x-increases from $x = 2$ to $x = 4$.

   Let f be a continuous function and $h > 0$. Simplify the quotient $\dfrac{f(4+h)-f(4)}{h}$ when
   (a) $f(x) = 3x + 2$.
   (b) $f(x) = x^2 - 3x + 5$.
2. Suppose $A\ (1, f(1))$ and $B(3, f(3))$ are points on the graph of $f(x) = x^2 + 1$. Find the slope of the line AB.
3. Let $h > 0$ and $A\ (x_0, f(x_0))$ and $B(x_o + h, f(x_0 + h))$ be points on the graph of a continuous function f. Find the slope of the line AB.

DEFINITION

Let $y = f(x)$ be a function

The expression $\dfrac{f(x_o+h)-f(x_o)}{h}$ is called the difference quotient of f at x_o with $h > 0$.

EXAMPLE 27

Find the difference quotient of the given function at the given point.

(a) $f(x) = 4x - 1$ at $x = 3$.

(b) $f(x) = x^2 + 7x + 8$ at $x = -2$.

Solution

(a) $\dfrac{f(3+h)-f(3)}{h}=\dfrac{4(3+h)-1-(4(3)-1)}{h}$

$=\dfrac{12+4h-1-12+1}{h}$

$=4.$

(b) $\dfrac{f(-2+h)-f(-2)}{h}=\dfrac{(-2+h)^2+7(-2+h)+8-(4+4+8)}{h}$

$=\dfrac{h^2-4h+4-14+7h+8+2}{h}$

$=\dfrac{h^2+3h}{h}=h+3.$

From this we see that, if h is close enough to 0, then 3 is the limiting value of $3 + h$.

Introduction

The concept limit is one of the most important topics of mathematics in learning differntiation and integration.

The limit of a function at a point is the value of the function near that point which may not be equal to its value at that point.

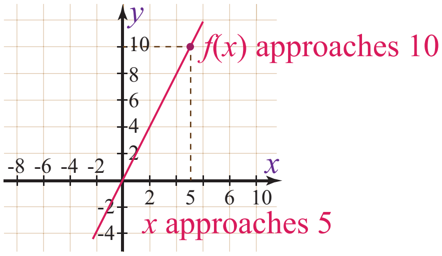
Let us use an example inorder to understand limts.

Let $f(x)=\dfrac{x^2-25}{x-5}$, Clearly, 5 is not in the domain of the fuction.
There is no corresponding value of y at $x = 5$.

However, for $x \neq 5$, $\dfrac{x^2-25}{x-5}=\dfrac{(x-5)(x+5)}{x-5}=x-5.$

That is we can not cancel the like factor $x - 5$ if $x = 5$, but we can cancel the like factor if $x \neq 5$.

Hence the value of the function at $x = 5$ which is $\dfrac{0}{0}$ is completely different from its value when x is near 5. The value of the function near $x = 5$ is 10 as shown in the figure below.



In this case we say the limit of the function as x approaches 5 is 10.

Activity 2

1. Identify the difference between the following paris of functions.

   (a) $f(x)=\dfrac{x^2}{x}$ and $g(x)=x$.

   (b) $f(x)=\dfrac{x^2-9}{x-3}$ and $g(x)=x+3$.

   (c) $f(x)=\dfrac{x^3-x}{x^2-x-2}$ and $g(x)=\dfrac{x^2+x}{x-2}$.

2. Let $f(x)=x^3-2x+7x+11$

   (a) Find $f(2)$.

   (b) Find the value of $f(x)$ as x approaches 2.

3. Let $f(x)=\dfrac{x^2-9}{x-3}$

   (a) Find $f(3)$.

   (b) Find the value of $f(x)$ as x approaches 3.

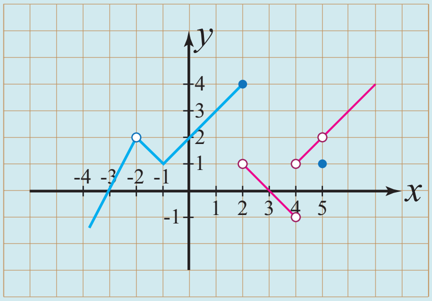
4. A figure given below is the graph of a function $y=f(x)$.

   Answer the following questions.

   (a) what is the domain of the function.

   (b) what is the value of $f(-1)$? $f(-2)$? $f(4)$? $f(5)$?

   (c) what is the value of $f(x)$ as x-approach -2? -1? 2? 4? 5?



Definition

Limit of a Function

Let $f(x)$ be defined on an open interval around x_o, except possibly at x_o itself. We say that the limit of $f(x)$ an x approaches is the number L, and write

$$\lim_{x\to x_o} f(x)=L.$$

Example 28

(a) $\lim_{x\to 3}(x+2)=3+2=5.$

(b) $\lim_{x\to -2}(x^2-3x-5)=(-2)^2-3(-2)-5=5.$

(c) $\lim_{x\to 4}1=1.$

The Limit Laws

If L, M, c and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} gf(x) = M$, then

1. Sum Rule:

   $\lim_{x\to c} (f(x) + g(x)) = L + M.$

2. Difference Rule:

   $\lim_{x\to c} (f(x) - g(x)) = L - M.$

3. Product Rule:

   $\lim_{x\to c} (f(x) \times g(x)) = LM.$

4. Constant Multiple Rule:

   $\lim_{x\to c} (kf(x)) = kL.$

5. Quotient Rule:

   $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M};\ M \neq 0.$

6. Power Rule: If m and n are relatively prime positive integers,

   $\lim_{x\to c} \left(f(x)\right)^{\frac{m}{n}} = L^{\frac{m}{n}}$, if n is even $L > 0$.

EXAMPLE 29

Use the limit rules to evaluate the following limits.

(a) $\lim_{x\to 2} (x^3 + 4x^2 - 5x + 7)$

(b) $\lim_{x\to -\frac{1}{2}} \frac{x^2 - 2x + 10}{1 - 6x^2}$

Solution

(a) $$\begin{aligned}\lim_{x\to 2} (x^3 + 4x^2 - 5x + 7) &= \lim_{x\to 2} x^3 + \lim_{x\to 2} 4x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7 \\ &= \left(\lim_{x\to 2} x\right)^3 + 4\left(\lim_{x\to 2} x\right)^2 - 5\lim_{x\to 2} x + 7 \\ &= 2^3 + 4(2)^2 - 5 \times 2 + 7 \\ &= 21.\end{aligned}$$

(b) $$\lim_{x\to -\frac{1}{2}} \frac{x^2 - 2x + 10}{1 - 6x^2} = \frac{\lim_{x\to -\frac{1}{2}} \left(x^2 - 2x + 10\right)}{\lim_{x\to -\frac{1}{2}} \left(1 - 6x^2\right)}$$

$$= \frac{\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 10}{1 - 6\left(-\frac{1}{2}\right)^2}$$

$$= \frac{-45}{2}.$$

EXAMPLE 30

Evaluate the following limits

(a) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2 - 3x + 2}$

(b) $\lim\limits_{x \to 1} \dfrac{x-2}{x^2 - 3x + 2}$

Solution

(a) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2 - 3x + 2} = \lim\limits_{x \to 2} \dfrac{x-2}{(x-2)\ (x-1)}$

$= \lim\limits_{x \to 2} \dfrac{1}{x-1}$

$= \dfrac{1}{2-1}$

$= 1.$

(b) $\lim\limits_{x \to 1} \dfrac{x-2}{x^2 - 3x + 2} = \lim\limits_{x \to 1} \dfrac{x-2}{(x-2)\ (x-1)}$

$= \lim\limits_{x \to 1} \dfrac{1}{x-1} = \dfrac{1}{0}$ which is undefined. This limit does not exist.

The Limit of Trigonometric Function

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

EXAMPLE 31

Evaluate the following limits

(a) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x}$

(b) $\lim\limits_{x \to 0} \dfrac{\sin x \cos x}{x}$

(c) $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sin(3x)}$ (d) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

Solution

(a) $$\lim_{x\to 0}\frac{\sin x^2}{x}=\lim_{x\to 0}\frac{x\sin x^2}{x\cdot x}=\lim_{x\to 0}x\cdot\frac{\sin x^2}{x^2}$$
$$=\left[\lim_{x\to 0}\right]\left[\lim_{x\to 0}\frac{\sin x^2}{x^2}\right]=0.1=0.$$

(b) $$\lim_{x\to 0}\frac{\sin x^2\cos x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\cos x=1\times 1=1.$$

(c) $$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}=\frac{\lim\limits_{x\to 0}\dfrac{2x\sin(2x)}{2x}}{\lim\limits_{x\to 0}\dfrac{3x\sin(3x)}{3x}}=\frac{\left[\lim\limits_{x\to 0}2x\right]\left[\lim\limits_{x\to 0}\dfrac{\sin(2x)}{2x}\right]}{\left[\lim\limits_{x\to 0}3x\right]\left[\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x}\right]}$$
$$=\left[\lim_{x\to 0}\frac{2\cancel{k}}{3\cancel{k}}\right]\left[\frac{1}{1}\right]=\frac{2}{3}.$$

(d) $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-\cos x}{x^2}.\frac{1+\cos x}{1+\cos x}$$
$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}.\frac{1}{1+\cos x}$$
$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}.\lim_{x\to 0}\frac{1}{1+\cos x}$$
$$=1\times\frac{1}{1+1}$$
$$=\frac{1}{2}.$$

Exercises

Evaluate the following limits

1. $\lim\limits_{x\to 2}(3x+7)$
2. $\lim\limits_{\to -}(5x^2-3x+4)$
3. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$
4. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x^2-9}$

5. $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-x-2}$

6. $\lim_{x\to 0}\frac{\sin(2x)}{x}$

7. $\lim_{x\to 0}\frac{\sin x\cos\ 2x}{2x}$

8. $\lim_{x\to 0}\frac{3x+\sin x}{4x}$

9. $\lim_{x\to 0}\frac{3\sin x-3x}{5x}$

10. $\lim_{x\to 0}\frac{1-\cos\ (5x)}{x^2}$

The Slope of a Curve

Definition

- A line is said to be tangent to the graph of a function $y = f(x)$ if it touches the graph at exactly one point in the neighborhood of the point on which the function is continuous. The point is said to be point of tangency.
- A line is said to be secant line to the graph of a function if it crosses the graph at two distinct points.
- The slope of the curve (or graph) at any point P is defined as the slope of the tangent line to the curve (or graph) at P.

Let $P(x, y)$ and $Q(x_1, x_1)$ be two distinct points on the curve $y = f(x)$ as shown in the figure.

The slope of the secant line passing through P and Q is given by $m_{sec} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$. As $x_1 \to x_0$, $Q \to P$ along the graph of f.

Thus the slope of the secant line approaches the slope of the tangent line at $P(x_0, y_0)$.

Therefore, the slope of the tangent line at $(x_o, f(x_0))$ is given by $m_{tan} = \lim_{x_1\to x_0}\frac{f(x_1)-f(x_0)}{x_1-x_0}$.

If $h = x_1 - x_2$ then, $x_1 = x_o + h$

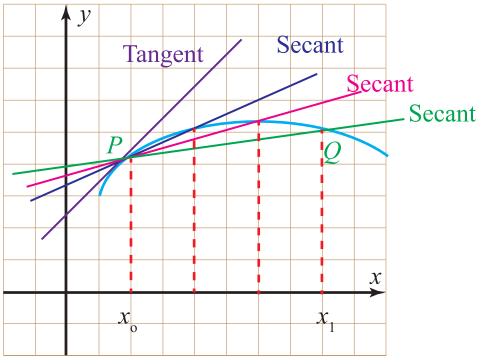


Figure 8.

Hence, $m_{tan} = \lim_{h\to o}\frac{f(x_o+h)-f(x_o)}{h}$.

Theorem: The slope of a line tangent to the graph of f at $(x_o, f(x_o))$ is $\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$.

Example 32

Find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 2$.

Solution

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{4}$$

$$= \lim_{h \to 0} \frac{(4+4h+h^2) - 4}{h}$$

$$= \lim_{h \to 0} \frac{4h+h^2}{h} = \lim_{h \to 0}(4+h) = 4.$$

Example 33

Find the slopes of $y = x^2 - x - 6$ at

(a) $x = 5$ (b) $x = 0$ (c) $x = -2$

Solution

Let us calcualte the slope at an arbitrary point at $(x, f(x))$. Then

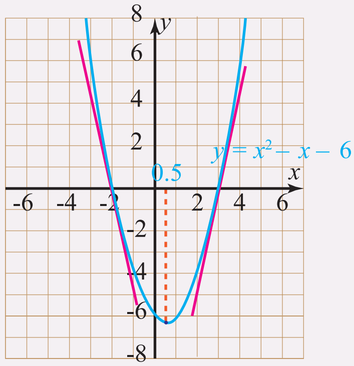
$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left((x+\mathrm{h})^2 - (x+h) - 6\right) - \left(x^2 - x - 6\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(2xh + h^2 - h\right)}{h}$$

$$= \lim_{h \to 0} \left(2x - 1 + h\right)$$

$$= 2x - 1.$$



This shows that, the slope of the curve $y = x^2 - x - 6$ at any point x is $2x - 1$. Thus,

A. The slope at $x = 5$ is $2(5) - 1 = 9$.

B. The slop at $x = 0$ is $2(0) - 1 = -1$.

C. The slop at $x = -2$ is $2(-2) -1 = -5$.

Exercises

For each of the following find the slope at the given values of x.

1. $y = 1 - x$ at

   (a) $x = 0$ (b) $x = 4$ (c) $x = c$, for a constant c.

2. $y = 4x^2 + 1$ at
   (a) $x = 0$ (b) $x = \frac{-1}{2}$ (c) $x = \frac{-1}{12}$
3. $y = 1 - x - 6x^2$ at
   (a) $x = 1$ (b) $x = \frac{-1}{6}$ (c) $x = \frac{-1}{12}$
4. $y = 2x^3 + 5$ at
   (a) $x = 0$ (b) $x = \sqrt{5}$ (c) $x = -\pi$
5. $y = x^4 - 5x^3 + 4x^2 + 6x + 1$ at
   (a) $x = 0$ (b) $x = -1$ (c) $x = \frac{1}{2}$
6. $y = -3x^3 + 2x^2 - x + 1$ at
   (a) $x = \frac{1}{9}$ (b) $x = 4$ (c) $x = -1$

For each of the following find the values of a, b, and c

7. $y = ax + b$ if the slope at $x = 0$ is 1.
8. $y = ax^2 + bx + c$ if the slopes at $x = 1$ is $\frac{17}{3}$ and at $x = -1$ is $\frac{13}{3}$.
9. $y = ax^2 - 4bx$ if the slope at $x = \frac{1}{4}$ is 0 and the slope at $x = \frac{1}{2}$ is 1.
10. $y = ax^3 + bx^2 + cx + d$ if the slope at $x = 1$ is 7, at $x = -1$ is 19 and at 0 is 7.

Differentiation

Definition

Let a be a number in the domain of a function f.

- If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists, then this limit is called the derivative of f at $x = a$

The derivative of f at a sis denoted by $f'(a)$; read $f'(a)$ as "f prime of a".

Hence, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. i.e $f'(a)$ is the limit of the difference quotient.

If $f'(a)$ exists, then we say that f has derivative at $x = a$ or f is differentiable at $x = a$.

Let $h = x - 1$, then $x = h + a$. If $x \to a$, then $h \to 0$.

Replacing x by $a + h$, gives $f'(a) = \lim_{x \to a} \frac{f(a+h) - f(a)}{h}$.

The process of finding the derivative is said to be differentiation.

EXAMPLE 34

Let $f(x) = 3$, f is a constant function. Find $f'(c)$.

Solution

$f(x) = 3$

$$\Rightarrow f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{3-3}{x-c} = 0.$$

Note:

- The derivative of a constant function at any real number is zero.
  i.e. If $f(x) = c$, then $f'(x) = 0$.

EXAMPLE 35

Find the derivative of $f(x) = 3x^2 + 5x + 8$ using the definition.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 + 5(x+h) + 8 - (3x^2 + 5x + 8)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 5x + 5h - 3x^2 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + 5h}{h}$$

$$= \lim_{h \to 0} (6x + 3h + 5)$$

$$= 6x + 5.$$

Alternative Notation for f' (x)

If $y = f(x)$, then we write $\frac{dy}{dx}$ to denote $f'(x)$

Thus, $\frac{dy}{dx} = f'(x)$.

EXAMPLE 36

Let $y = x^3 + 1$, find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 + 1 - (x^3 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - (x^3 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{\to} \left(3x^2 + 3xh + h^2\right)$$

$$= 3x^2.$$

The Derivative of $f(x) = x^n$ or $y = x^n$

The Power Rule

Activity 3

Using the definition of derivatives, fill in the following table.

$f(x)$	$f'(x)$
x	
x^2	
x^3	
x^4	

The Power Rule

Let $f(x) = x^n$, then $f'(x) = nx^{n-1}$ where n is an integer.

If $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$.

Function $y = f(x)$	**Derivative** $\frac{dy}{dx} = f'(x)$		**Function** $y = f(x)$	**Derivative** $\frac{dy}{dx} = f'(x)$
x	1		$\frac{1}{x}\left(\text{or } x^{-1}\right)$	$-x^{-2} = \frac{1}{x^2}$
$2x^2$	$4x$		$\frac{4}{x^2}$	$4(-2)x^{-3} = \frac{-8}{x^3}$
$5x^3$	$15x^2$		x^{-7}	$-8x^{-8}$
$7x^6$	$42x^5$		$-3x^{-5}$	$15x^{-6}$
x^9	$9x^8$		$\frac{\sqrt{5}}{2x^4}$	$\frac{\sqrt{5}}{2}\left(-4x^{-5}\right) = \frac{-2\sqrt{5}}{x^5}$
x^{20}	$20x^{19}$			

You have seen the equation of a line tangent to the graph of $y = f(x)$ at $(x_o, f(x_o))$ is

$y - y_o = m(x - x_o)$ where $\lim\limits_{\to} \frac{f(x_o + h) - f(x_o)}{}$

$\Rightarrow y - y_o = f'(x_o)(x - x_o)$

$\Rightarrow y = f'(x_o)(x - x_o) + y_o.$

Example 37

Find the equation of the line tangent to the graph of

(a) $f(x) = 3x^2 - 9x + 1$ at $(1, -5)$

(b) $f(x) = 4x^5 - 2x^2 + 3$ at $(0, 3)$

Solution

(a) $f'(x) = 6x - 9$

$\Rightarrow f'(1) = -3$

$y = -3(x - 1) - 5$

$\Rightarrow y = -3x - 2.$

(b) $f'(x) = 20x^4 - 4x$

$\Rightarrow f'(x) = 0$

$\Rightarrow y = 3.$

Example 38

Find the equation of the graph of the line tangent to the graph of

(a) $f(x) = \frac{4}{x^5}$ at $(1, 4)$

(b) $f(x) = \frac{1 - 2x^3}{x^2}$ at $\left(\frac{-1}{2}, 5\right)$

Solution

(a) $f(x) = 4(x^{-5})$

Thus, the slope is $f'(x) = -20x^{-6}$

$\Rightarrow f'(1) = -20.$

Therefore, the equation of the tangent line is

$y = -20(x - 1) + 4$

$y = -2x + 24.$

(b) $f(x) = \frac{1 - 2x^3}{x^2}$

$= \frac{1}{x^2} - 2x$

$$f'(x) = \frac{-2}{x^3} - 2.$$

Thus, the slope is $f'\left(-\frac{1}{2}\right) = 14.$

Therefore, the equation of the tangent lien is

$$y = 14\left(x + \frac{1}{2}\right) + 5$$

$$y = 14x + 12.$$

The Derivatives of Trigonometric Functions.

The Derivative of Sine Function

If $y = \sin x$, then $\frac{dy}{dx} = \cos x$

ACTIVITY 4

1. Let $f(x) = \sin x$, find $\lim_{h \to 0} \frac{\sin h - \sin 0}{h}$
2. Let $f(x) = \cos x$, find $\lim_{h \to 0} \frac{\cos h - \cos 0}{h}$

EXAMPLE 39

Find the equation of the line tangent to the graph

(a) $f(x) = \frac{1}{2} \sin x$ at $(\pi, 0)$.

(b) $f(x) = 3 - 2 \cos x$ at $(0, 1)$.

Solution

(a) $f(x) = \frac{1}{2} \sin x$

$f'(x) = \frac{1}{2} \cos x$

Thus, the slope $f'(\pi) = \frac{1}{2} \cos \pi = -\frac{1}{2}.$

Therefore, the equation of the tangent line is $y = -\frac{1}{2}(x - \pi).$

(b) $f(x) = 3 - 2 \cos x$

$f'(x) = 0 - 2(-\sin x)$

$= 2 \sin x.$

Thus, the slope is $f'(0) = 2 \sin 0 = 0$.

Therefore, the equation of the tangent line is $y = 1$.

Example 40

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 5x + 4$ which has a slope of $\frac{1}{2}$.

Solution

$f(x) = 3x^2 - 5x + 4$

$\Rightarrow f'(x) = 6x - 5,$ given $f'(x) = \frac{1}{2}$.

$\Rightarrow 6x - 5 = \frac{1}{2} \Rightarrow x = \frac{11}{12}$.

The equation of the tangent line is: $y = \frac{1}{2}x + \frac{71}{48}$.

Example 41

Find the equation of the line tangent to the graph of $f(x) = x^2 - 5x + 6$ at the point where the graph intersects,

(a) the x-axis (b) the y-axis (c) the line $y = 2$.

Solution

$f(x) = x^2 - 5x + 6$

$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}(x^2 - 5x + 6) = 2x - 5$

$\Rightarrow$ The slope function, $f'(x) = 2x - 5$.

(a) $x^2 - 5x + 6 = 0$

$\Rightarrow x = 2$ or $x = 3$ (the graph intersects the x axis at $x = 2$ or $x = 3$)

(i) The equation of the tangent lie at $x = 2$ is:

$y - f(2) = f'(2)(x - 2)$

$\Rightarrow y - 0 = -1(x - 2)$

$\Rightarrow y = 2 - x.$

(ii) The equation of the tangent at $x = 3$ is:

$y - f(3) = f'(3)(x - 3)$

$\Rightarrow y - 0 = 1(x - 3)$

$\Rightarrow y = x - 3.$

Note:

- The tangents in (i) and (ii) are perpendicular.

(b) The graph intersects the y-axis at $x = 0$

$y - f(0) = f'(0)(x - 0)$

$\Rightarrow y - 6 = -5x \Rightarrow y = 6 - 5x.$

(c) $x^2 - 5x + 6 = 2 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow x = 1$ or $x = 4$.

(i) The equation of the tangent line at $x = 1$ is:

$y - f(1) = f'(1)(x - 1)$

$\Rightarrow y - 2 = -3(x - 1)$

$\Rightarrow y = 5 - 3x.$

(ii) The equation of the tangent line at $x = 4$ is:

$y - f(4) = f'(4)(x - 4)$

$\Rightarrow y - 2 = 3(x - 4) \Rightarrow y = 3x - 10.$

Exercises

For each of the following find $f'(c)$ for the given value of c by the definition of derivatives .

1. $f(x) = -x$; $c = 1$
2. $f(x) = \frac{1}{3}x$; $c = 0$
3. $f(x) = 5x + 1$; $c = \frac{-1}{7}$
4. $f(x) = 4 - \frac{3}{2}x$; $c = \sqrt{3}$
5. $f(x) = \frac{x+5}{12}$; $c = \pi$
6. $f(x) = x^2$; $c = \frac{-1}{\pi}$
7. $f(x) = x^2 - 3x + 11$; $c = -\frac{1}{5}$
8. $f(x) = \sqrt{x}$; $c = \frac{1}{9}$
9. $f(x) = \sqrt[5]{x}$; $c = 32$
10. $f(x) = \frac{1}{\sqrt[3]{x^4}}$; $x = 1$

11. $f(x) = -2\sin x;\ c = 0$

12. $f(x) = \frac{1}{5}\cos x - 1;\ x = \frac{\pi}{2}$

13. $f(x) = \sin x; c = \frac{\pi}{4}$

14. Find the equation of the line tangent to the graph of $f(x) = \sin x$ at $x = 0$.

15. Find the equation of the line tangent to the graph of $f(x) = 1 - 3x + 2x^2$ at $x = 1$.

16. Find the equation of the tangent to the graph of $f(x) = x^2 - x^3$ at $x = 2$.

17. Find the equation of the tangent to the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 + x + 7$ *at*:

(a) $x = \frac{-3}{4}$ (b) $x = 1$

Differentiation of Combinations of Functions

Derivative of a Sum

If $y = f(x) + g(x)$, then $\frac{dy}{dx} = f'(x) + g'(x)$.

EXAMPLE 42

Let $y = 3x^2 + 4$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 6x.$$

Derivative of a Difference

If $y = f(x) - g(x)$, then $\frac{dy}{dx} = f'(x) - g'(x)$

EXAMPLE 43

Let $y = x^3 - 2x^2 + 1$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 3x^2 - 4x.$$

Example 44

Let $f(x) = 6x^3 + 5x^2 + 4$. Find $(6f(x))'$

Solution

$(6f(x))' = 6f'(x) = 6(6x^3 + 5x^2 + 4)' = 6(18x^2 + 10x)$.

Example 45

Let $y = \sin x - 2\cos x$, find $\frac{dy}{dx}$

Solution

$\frac{dy}{dx} = \cos x - 2(-\sin x) = \cos x + 2\sin x$.

Derivative of a Product

If $y = f(x)\,g(x)$, then $\frac{dy}{dx} = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$

Example 46

Let $f(x) = (4x + 5)(3 - 7x)$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = (4x + 5)'(3 - 7x) + (4x + 5)(3 - 7x)'$$
$$= 4(3 - 7x) + (4x + 5) \times (-7) = -56x - 23.$$

Example 47

Find the derivative of $f(x) = (x^2 + 1)(2x^3 - 5x + 11)$ with respect to x.

Solution

the derivatives of the functions.

$h(x) = x^2 + 1$ and $g(x) = 2x^3 - 5x + 11$ are

$h'(x) = 2x$ and $g'(x) = 6x^2 - 5$ respectively.

$f(x) = h(x)\,g(x)$

$\Rightarrow f'(x) = h'(x)\,g(x) + h(x)\,g'(x)$

$= 2x(2x^3 - 5x + 11) + (x^2 + 1)(6x^2 - 5) = 10x^4 - 9x^2 + 22x - 5.$

Note that $f(x) = (x^2 + 1)(2x^3 - 5x + 11) = 2x^5 - 3x^3 + 11x^2 - 5x + 11$

$\Rightarrow f'(x) = 10x^4 - 9x^2 + 22x - 5.$

EXAMPLE 48

Differentiate $f(x) = (x^2 + 5x + 1)(3x^3 - 4x^2 + 7)$ with respect to x.

Solution

$$f(x) = (x^2 + 5x + 1)(3x^3 - 4x^2 + 7)$$

$$\Rightarrow f'(x) = (x^2 + 5x + 1)'(3x^3 - 4x^2 + 7) + (x^2 + 5x + 1)(3x^3 - 4x^2 + 7)'$$

$$= (2x + 5)(3x^3 - 4x^2 + 7) + (x^2 + 5x + 1)(9x^2 - 8x)$$

$$= 15x^4 + 44x^3 - 51x^2 + 6x + 35.$$

EXAMPLE 50

Let $y = \sin x \cos x$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \cos x(\cos x) + \sin x(-\sin x) = \cos^2 x - \sin^2 x.$$

Derivative of the Quotient

If $y = \frac{f(x)}{g(x)}$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)\,g'(x)}{(g(x))^2}$$

EXAMPLE 49

Differentiate $h(x) = \frac{x}{x+1}$ with respect to x.

Solution

Let $f(x) = x$ and $g(x) = x + 1$. Then $h(x) = \frac{f(x)}{g(x)}$.

Therefore, $h'(x) = \frac{f'(x)g(x) - f(x)\,g'(x)}{(g(x))^2}$.

But $f'(x) = 1$ and $g'(x) = 1$.

Therefore, $h'(x) = \frac{1(x+1) - 1(x)}{(g(x))^2}$.

But $f'(x) = 1$ and $g'(x) = 1$.

Therefore, $h'(x) = \frac{1(x+1) - 1(x)}{(x+1)^2} = \frac{1}{(x+1)^2}$.

Example 51

Let $y = \tan x$, find $\frac{dy}{dx}$

Solution

$y = \tan x = \frac{\sin x}{\cos x}$.

Thus, $\frac{dy}{dx} = \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x$.

Example 52

Let $f(x) = \frac{2x-3}{x^2+4}$, find $f'(x)$

Solution

Let $h(x) = 2x - 3$ and $g(x) = x^2 + 4$. Then $f(x) = \frac{h(x)}{g(x)}$

$f'(x) = \frac{h'(x)\,g(x) - h(x)\,g'(x)}{(g(x))^2}$ with $h'(x) = 2$ and $g'(x) = 2x$.

Therefore,

$$f'(x) = \frac{2(x^2+4) - (2x-3)(2x)\; 2x^2 + 8 - (4x^2 - 6x)}{\left(x^2+4\right)^2} = \frac{-2x^2 + 6x + 8}{\left(x^2+4\right)^2}.$$

Exercises

For each of the following compute (Questions 1 - 5)

(a) $(f+g)'(x)$ (c) $(f-g)'(x)$

(b) $(fg)'(x)$ (d) $\left(\frac{f}{g}\right)(x)$

1. $f(x) = 3x - 1$ and $g(x) = 5 - 3x$.
2. $f(x) = x^2$ and $g(x) = 7x + 1$.

3. $f(x) = 4x^3 + 5x + 2$ and $g(x) = x^2 + 1$.
4. $f(x) = \sin x$ and $g(x) = \cos x$.
5. $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

Consider the function $f(x) = x^2$. When f is differentiated with respect to x we have seen that $f'(x) = 2x$.

In this unit we are going to find a function whose derivative is given. i.e. given $f'(x)$, we want to know $f(x)$ (*its anti-derivative*).

Also, we have seen that different functions may have the same derivative.

For instance, all of the following functions have the same derivative.

A. $f(x) = x^2 \Rightarrow f'(x) = 2x$

B. $f(x) = x^2 - 9 \Rightarrow f'(x) = 2x$

C. $f(x) = x^2 + \pi \Rightarrow f'(x) = 2x$

D. $f(x) = x^2 + c \Rightarrow f'(x) = 2x$; c is a constant.

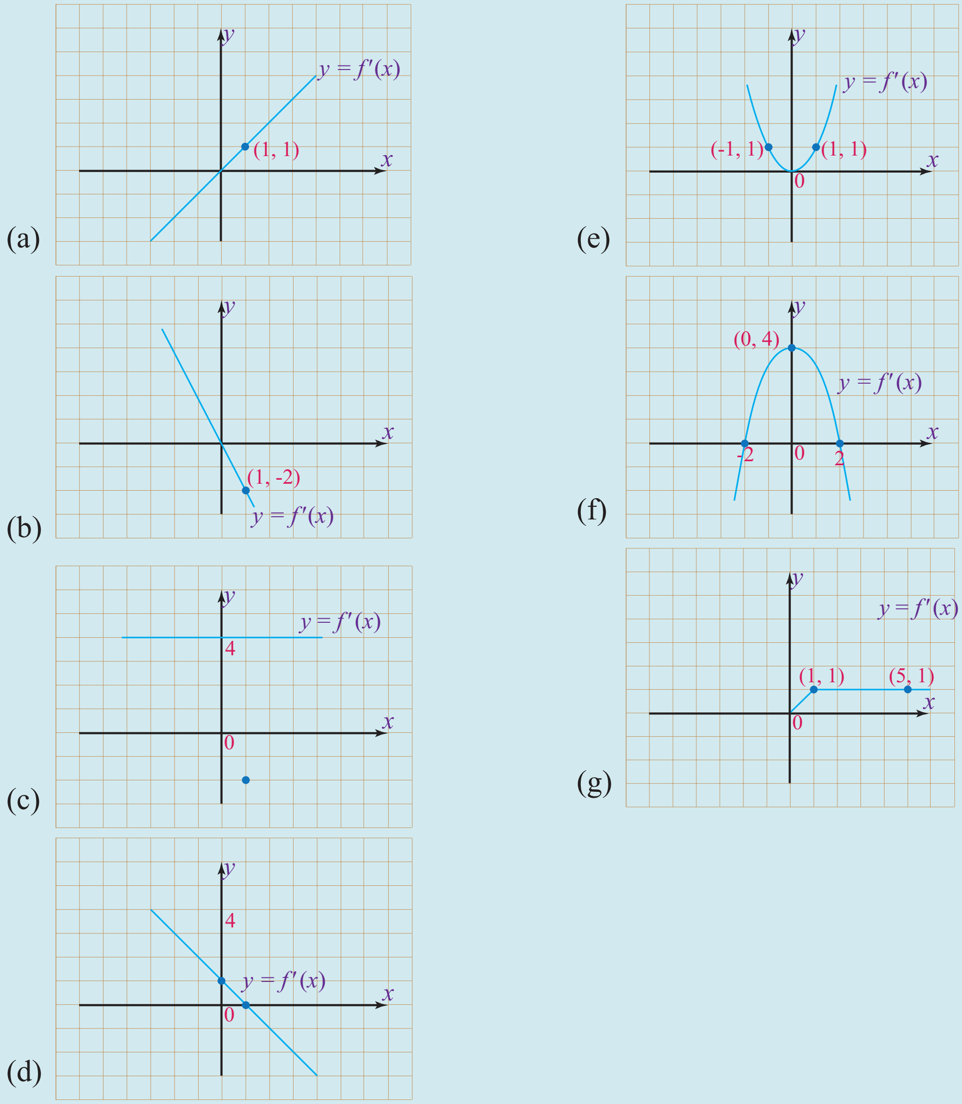
Thus, the functions $f(x) = x^2$, $f(x) = x^2 - 9$, $f(x) = x^2 + \pi$ are particular examples of the ant-derivatives of $f'(x) = 2x$ and $f(x) = x^2 + c$ is the general case.

ACTIVITY 5

1. The first column of the following table contains the derivative of a function f. Fill in the second column of the table by a formula for $f(x)$.

$f'(x)$	$f(x)$
0	
2	
$3x$	
$2x + 1$	
$\sin x$	
$\cos x$	
x^2	
$x^2 - x + 2$	

2. Given the graph of $y = f'(x)$, sketch the graph of f.



Definition

- The process of finding a function $f(x)$ from its derivative $f'(x)$ is called integration (or anti differentiation).
- The set of anti-derivatives of $f(x)$ is called the indefinite integral of $f(x)$.

Integration reverses the operation of differentiation. Hence the integral (or anti-derivative) of $f'(x)$ is $f(x) + c$; where c is a constant and c is called the constant of integration. c can be any real number including 0.

Note that $(f(x) + c)' = f'(x) + (c)' = f'(x) + 0 = f'(x)$.

Notation

- The integral of $f(x)$ is denoted as:
  $\int f(x)\,dx$.

The symbol $\int$ is called an integral sign. $f(x)$ is called the integrand.

$\int f(x)\,dx$ means "the integral of $f(x)$ with respect to x" and x is called the variable of integration. If a function has an integral, then it is said to be integrable.

Note

If $F(x)$ is an anti-derivative of $f(x)$, then $\int f(x)\,dx = F(x) + c$.

Theorem

- If $F(x)$ and $G(x)$ are anti derivatives of the function $f(x)$ in the interval $[a, b]$, then $F(x) = G(x) + c$ for all $x \in [a, b]$ for a constant c.
  ($F(x)$ and $G(x)$ differ by a constant)

EXAMPLE 53

(a) $5\,dx = 5x + c$, since $(5x + c)' = 5$.

(b) $0\,dx = c$, because $(c)' = 0$

(c) $7\,dy = 7y + 6$ (The integral of 7 with respect to y is $7y + c$)

(d) $2t\,dt = t^2 + c$, (The integral of $2t$ with respect to t is $t^2 + c$)

(e) $\int 5x\,dx = \frac{5x^2}{2} + c$, since $\left(\frac{5x^2}{2} + c\right)' = 5x$.

EXAMPLE 54

Let $f'(x) = 3x$. Find $f(x)$ if $f(0) = 2$.

Solution

$f'(x) = 3x$

$\Rightarrow f(x) = \int f'(x)\,dx = \int (3x)\,dx$

$= \frac{3x^2}{2} + c.$

But $f(0) = 2$.

Hence $\frac{3(0)^2}{2}+c=2 \Rightarrow c=2.$

Therefore, $f(x)=\frac{3x^2}{2}+2.$

Example 55

Let $f'(x) = x$. Find $f(x)$ if $f(-1) = 7$.

Solution

$f'(x) = x$

$\Rightarrow f(x)=\int f'(x)\,dx=\int x\,dx=\frac{x^2}{2}+c.$

But $f(-1) = 7$, $\frac{(-1)^2}{2}+c=7$.

$\Rightarrow \; c = 7 - \frac{1}{2} = \frac{13}{2}.$

Therefore, $f(x) = \frac{x^2}{2} - \frac{13}{2}.$

Example 56

(a) $\int a\,dx = ax + c$ because $(ax + c)' = a$

(b) $\int x^2\,dx=\frac{x^3}{3}+c$, since $\left(\frac{x^3}{3}+c\right)'=\left(\frac{x^3}{3}\right)'+(c)'=\frac{3x^2}{3}+0=x^2.$

(c) $\sin x\,dx = -\cos x + c$,

Since $(-\cos x + c)' = (-\cos x)' + (c)' = \sin x.$

(d) $\int \cos x\,dx = \sin x + c.$

Example 57

Let $f'(x) = \sin x$. Find $f(x)$ if $f(0) = 3$.

Solution

$f'(x) = \sin x$

$\Rightarrow \int f'(x)\,dx = \sin x\,dx + c$

$\Rightarrow f(x) = -\cos x + c$

But $f(0) = 3 \Rightarrow f(0) = -\cos(0) + c$

$\Rightarrow 3 = -1 + c \Rightarrow \; c = 4.$

Therefore, $f(x) = 4 - \cos x.$

Example 58

(a) $\int (7x-1)\,dx = \frac{7x^2}{2} - x + c$

Because $\left(\frac{7x^2}{2} - x + c\right)' = \left(\frac{7x^2}{2}\right)' - (x)' + (c)' = 7x - 1$

Therefore $\int (7x-1)\,dx = \frac{7x^2}{2} - x + c.$

(b) $\int 3x^3\,dx = \frac{3x^4}{4} + c.$

Exercises

Find each of the following integrals

1. $\int 3\,dx$
2. $\int \sqrt{7}\,dx$
3. $\int (-x)\,dx$
4. $\int 5x\,dx$
5. $\int (4x+1)dx$
6. $\int \pi x\,dx$
7. $\int (x^2 + x + 1)\,dx$
8. $\int \sin(-x)\,dx$
9. $\int ex\,dx$
10. $\int 6x^5\,dx$

Theorem: (The Power Rule)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c; n \neq -1$$

Proof: $\left(\frac{x^{n+1}}{n+1} + c\right)' = \left(\frac{x^{n+1}}{n+1}\right)' + (c)'$

$$= \frac{1}{n+1}(n+1)x^{(n+1-1)} + 0 = x^n.$$

Example 59

(a) $\int x\,dx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

(b) $\int x^2\,dx = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$

(c) $\int x^3\,dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$

(d) $\int x^9\,dx = \frac{x^{9+1}}{9+1} + c = \frac{x^{10}}{10} + c$

(e) $\int x^{105}\,dx = \frac{x^{105+1}}{105+1} + c = \frac{x^{106}}{106} + c$

Exercises

Using the rule $\int x^n dx = \frac{x^{n+1}}{n+1} + c; n = 1, 2, 3, \ldots$ find the following inegerals

1. $\int x\,dx$
2. $\int 3\,x^2\,dx$
3. $\int -4x^3\,dx$
4. $\int 10x^4\,dx$
5. $\int 4x^5\,dx$
6. $\int -14x^6\,dx$
7. $\int 24x^7\,dx$
8. $\int x^{18}\,dx$
9. $\int -15x^{29}\,dx$
10. $\int x^{70}\,dx$
11. $\int 18x^{505}\,dx$
12. $\int 12x^{1121}\,dx$

Theorem:

Let f and g be two continuous functions, then

1. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
2. $\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$
3. $\int kf(x)\,dx = k \int f(x)\,dx$ for a constant k.

Example 60

Find $\int (x+1)\,dx$

Solution

$\int (x+1)dx = x\,dx + (1)\,dx$

$= \frac{x^2}{2} + x + c$, since $c_1 + c_2$ is a constant c.

Note that $\frac{d}{dx}\left(\frac{x^2}{2} + x + c\right) = \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}x + \frac{d}{dx}c = \frac{2x}{2} + 1 + 0 = x + 1$

Therefore, $\int (x+1)\,dx = \frac{x^2}{2} + x + c$.

Example 61

Find $\int (2x-1)\, dx$.

Solution

$$\int (2x-1)dx = 2x\, dx - 1\, dx = \frac{2x^2}{2} + c_1 - x + c_2 = x^2 - x + c.$$

Find $\int (x^2 - x)dx$.

Solution

$$(x^2 - x)\, dx = x^2\, dx - x\, dx$$

$$= \frac{x^3}{3} + c_1 - \frac{x^2}{2} + c_2 = \frac{x^3}{3} - \frac{x^2}{2} + c_1 + c_2 = \frac{x^3}{3} - \frac{x^2}{2} + c.$$

Example 62

Find $\int (x^3 - x^2 + x - 1)dx$.

Solution

$$\int (x^3 - x^2 + x - 1)\, dx = x^3\, dx - x^2\, dx + x\, dx - dx$$

$$= \frac{x^4}{4} + c_1 - \frac{x^3}{3} + c_2 + \frac{x^2}{2} + c_3 - x + c_4$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + c_1 + c_2 + c_3 + c_4$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + c.$$

Example 63

Find $\int \left(3x^5 - \frac{1}{4}x^4 + \sqrt{2}\, x^3 + 17x\right) dx.$

Solution

$$\int \left(3x^5 - \frac{1}{4}x^4 + \sqrt{2}x^3 + 17x\right) dx = \int 3x^5 dx - \int \frac{1}{4}x^4 dx + \int \sqrt{2}x^3 dx + \int 17x\, dx$$

$$= 3\int x^5 dx - \frac{1}{4}\int x^4 dx + \sqrt{2}\int x^3 dx + 17\int x dx$$

$$= \frac{3x^6}{6} + c_1 - \frac{1}{4}\frac{x^5}{5} + c_2 + \frac{\sqrt{2}x^4}{4} + c_3 + \frac{17x^2}{2} + c_4$$

$$= \frac{x^6}{2} - \frac{x^5}{20} + \frac{\sqrt{2}x^4}{4} + \frac{17x^2}{2} + c_1 + c_2 + c_3 + c_4.$$

$$= \frac{x^6}{2} - \frac{x^5}{20} + \frac{\sqrt{2}x^4}{4} + \frac{17x^2}{2} + c.$$

Example 64

Find $\int \left(x^{31} - \frac{1}{5}x^{29} + 17 \right) dx$

Solution

$$\int \left(x^{31} - \frac{1}{5}x^{29} + 17 \right) dx = \int x^{31} dx - \int \frac{1}{5}x^{29} dx + \int 17 dx$$

$$= \frac{x^{32}}{32} + c_1 - \frac{1}{5}\frac{x^{30}}{30} + c_2 + 17x + c_3$$

$$= \frac{x^{32}}{32} - \frac{x^{30}}{150} + 17x + c.$$

Exercises

Find the integral for each of the following

1. $\int (x+3)\, dx$
2. $\int (2x-5)\, dx$
3. $\int (x+1)(x-3)\, dx$
4. $\int (x^2 - 2x)\, dx$
5. $\int (4x^2 + 5x + 1)\, dx$
6. $\int (\sqrt{2}x^5 - 4x^3 + 5x^2 + 2x - 10)\, dx$
7. $\int (ax^2 + bx^3 + cx^4 + dx + e)\, dx$
8. $\int \left(\left(\frac{1}{4}x^3 - 6x^2 \right) + (5x^4 - 12) \right) dx$
9. $2\int 4x\, dx - 4\int 5x\, dx + 12\int 11x\, dx$
10. $\int (x^{95} - x^{59} + 1)\, dx$
11. $\int (\cos x - \sin x)\, dx$
12. $\int (x^3 - x^4)\, dx$
13. $\int (\sin x - x + 1)\, dx$
14. $\int x^{2n} dx - \int x^{2n-1}\, dx$
15. $\int (\sin x - \cos x + x^2)\, dx$

Definition

- Let f be continuous on $[a, b]$. The definite integral of f from a to b is denoted by $\int f(x)dx$.

Read $\int f(x)dx$ as "The integral of f with respect to x from a to b".
The numbers a and b are called the limits of integration. a is called the lower limit of integration and b is called the upper limit of integration.

Theorem: (Fundamental Theorem of Calculus)

- If f is continuous on $[a, b]$ and if F is an anti-derivative of f, that is $F'(x) = f(x)$, $x \in [a, b]$, then $\int f(x)dx = F(b) - F(a)$.

Example 65

Evaluate the following definite integrals.

(a) $\int_1^2 xdx$ (c) $\int_{-4}^7 3x\,dx$

(b) $\int_3^5 (x+4)dx$ (d) $\int_{0.7}^{6.7} (4x-5)\,dx$

Solution

(a) $\int_1^2 xdx$

$\int x\,dx = \frac{x^2}{2} + c$; the definite integral uses the anti-derivatives of x.

$$\text{Therefore, } \int_1^2 x\,dx = \left(\frac{x^2}{2} + c\right)\Bigg|_1^2 = \left(\frac{x^2}{2} + c\right)\Bigg|_{x=2} - \left(\frac{x^2}{2} + c\right)\Bigg|_{x=1}$$

$$= \left(\frac{4}{2} + c\right) - \left(\frac{1}{2} + c\right) = \frac{3}{2}.$$

Note that the constant c is not needed.

$$\text{Therefore, } \int_1^2 x\,dx = \frac{x^2}{2}\Bigg|_1^2 = \frac{(2)^2}{2} - \frac{(1)^2}{2} = \frac{3}{2}.$$

Note that the indefinite integral of x, $\int x\,dx$ is a function $f(x)=\frac{x^2}{2}+c$.

However, the definite integral $f(x)=\int_1^2 x\,dx = \frac{3}{2}$ is a number and it contians no variable x, no constant c.

(b) $\int_3^5 (x+4)dx = \left(\frac{x^2}{2} + 4x\right)\Big|_3^5$

$$= \left(\frac{(5)^2}{2} + 4(5)\right) - \left(\frac{3^2}{2} + 4(3)\right)$$

$$= \left(\frac{25}{2} + 20\right) - \left(\frac{9}{2} + 12\right) = 16.$$

(c) $\int_{-4}^7 3x\,dx = 3\int_{-4}^7 3x\,dx = 3\left(\frac{x^2}{2}\right)\Big|_{-4}^7 = \frac{99}{2}$.

(d) $\int_{0.7}^{6.7} (4x-5)\,dx = (2x^2 - 5x)\Big|_{0.7}^{6.7}$

$$= 2(6.7)^2 - 5(6.7) - (2\,(0.7)^2 - 5(0.7))$$

$$= 58.8.$$

Example 66

Evaluate the following integrals

(a) $\int_{-2}^3 (x^2 - 1)\,dx$

Solution

$$\int_{-2}^3 (x^2-1)\,dx = \left(\frac{x^3}{3} - x\right)\Bigg|_{-2}^3$$

$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{(-2)^3}{3} - (-2)\right) = 6 + \frac{8}{3} - 2 = \frac{18+8-6}{3} = \frac{20}{3}.$$

(b) $\int_0^\pi \sin x\,dx$

Solution

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi - [\cos\pi - \cos 0] = -[-1-1] = 2.$$

(c) $\int_7^{11} 4\, dx$

Solution

$$\int_7^{11} 4\, dx = 4x\Big|_4^{11} = 4(11) - 4(7) = 16.$$

Note

$\int_a^b c\, dx = cx\Big|_a^b = c(b-a)$ for a constant c.

Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions in $[a, b]$.

i.e $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are defined. Then

1. $\int_a^b (f+g)(x)\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b (f-g)(x)(dx) = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$
3. $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$
4. If $a \le c \le b$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
5. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
6. $\int_a^a f(x)dx = 0$.

Exercises

Evaluate each of the following

1. $\int_5^3 3\, dx$
2. $\int_{-100}^{70} 0\, dx$
3. $\int_{-5}^{4} x\, dx$
4. $\int_m^n c\, dx$
5. $\int_5^7 \sqrt{3}\, x\, dx$
6. $\int_{\frac{-1}{2}}^{3} 3x\, dx$
7. $\int_{-5}^{5} x^2\, dx$
8. $\int_{-\sqrt{2}}^{\frac{1}{3}} (2x+1)\, dx$
9. $\int_0^{10} \left(1-\frac{1}{2}x\right) dx$
10. $\int_2^{21} \left(6-x^2\right) dx$
11. $\int_{-3}^{-1} \left(\frac{1}{2}x+4x^2-2\right) dx$

12. $\int_{-11}^{0} (2x+1)^2 \, dx$

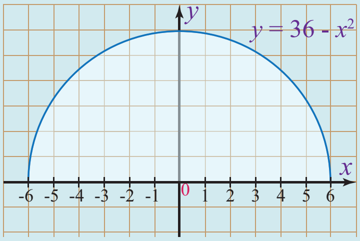
13. $\int_{\frac{-\pi}{2}}^{\pi} \sin x \, dx$

14. $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$

Area Under a Curve

Activity 6

Approximating the area of the region under graph of $y = 36 - x^2$ in figure shown below.



1. Copy and fill in the following table.

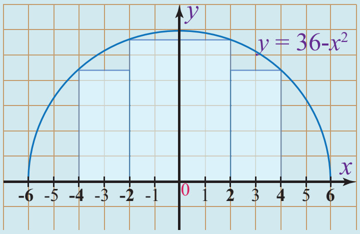
x	-4	-2	0	2	4
$y = 36 - x^2$					

Approximate the area of the above region by dividing the region into four rectangular strips as shown below.

(a) Find the width and height of each rectangular strips.

(b) Find area of each rectangular strips.

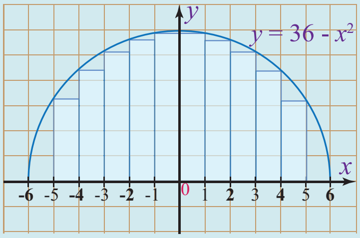
(c) Find the sum of the areas of the rectangular strips.



2. In order to get a better approximation of the region, we increase the number of the rectangular strips as shown below.

copy and fill in the following table.

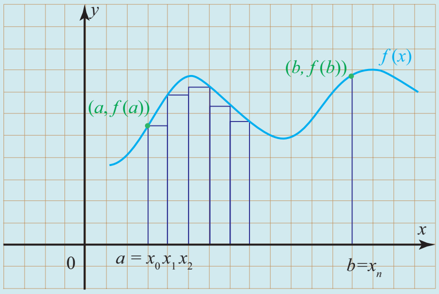
Use the following table to determine the width and height of the rectangular strips. Find the sum of the areas of the rectangular strips



x	-5	-4	-2	-1	0	1	2	3	4	5
$y = 36 - x^2$										

3. What will be the difference between the area of the region and the sum of the areas of the strips as the width of the rectangles approaches to zero?

For instance, if the width of each rectangle is $\frac{1}{2}$, then the area of each rectangle is $\frac{1}{2}(36 - x^2)$ as shown in the following table.



x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$
Height $36 - x^2$	36	$\frac{143}{4}$	35	$\frac{135}{4}$	32	$\frac{119}{4}$	27	$\frac{95}{4}$	20	$\frac{63}{4}$	11	$\frac{23}{4}$
Area $\frac{1}{2}(36 - x^2)$	18	$\frac{143}{8}$	$\frac{35}{2}$	$\frac{135}{8}$	16	$\frac{119}{8}$	$\frac{27}{2}$	$\frac{95}{8}$	10	$\frac{63}{8}$	$\frac{11}{2}$	$\frac{23}{8}$

The sum of the area of the rectangles is

4. $2\left(18 + \frac{143}{8} + \frac{35}{2} + \frac{135}{8} + 16 + \frac{119}{8} + \frac{27}{2} + \frac{95}{8} + 10 + \frac{63}{8} + \frac{11}{2} + \frac{23}{8}\right) = 249.25$

As the number of the rectangles increases, the sum of the areas of the rectangles get closer and closer to the area of the region. Eventually, the area of the region will be determined using integration.

In general, we can find the area of a region under a curve using integrals as follows.

Consider the graph of $y = f(x)$, the lines $x = a$ and $x = b$.

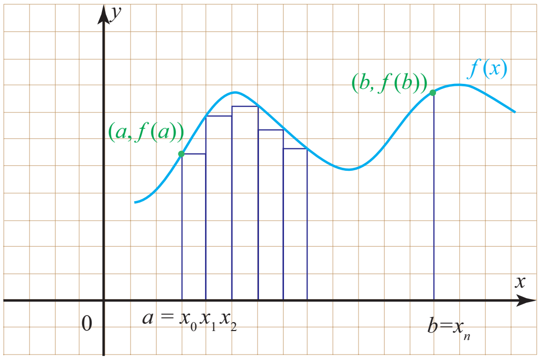


Figure 10.

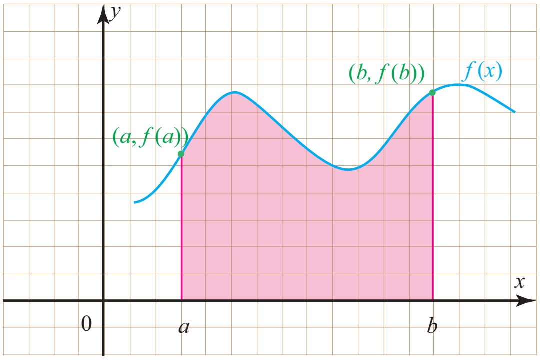


Figure 9.

We want to find the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

Divide the region into vertical strips as shown in the figure.

The sum of the areas of these rectangles will approximately give the area of the region.

As the strips get thinner and thinner, the accuracy of the approximation of the area can increase.

Therefore, if the areas of all strip are summed from $x = a$ to $x = b$, it gives the exact area of the region.

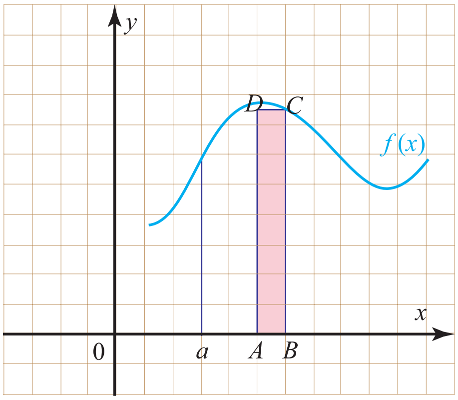


Figure 11.

Area using Integrals

The area of a region under the graph of a continuous function between $x = a$ by $x = b$ is given by $A = \int_a^b f(x)\,dx$

Example 67

Find the area bounded by the line $x = 1$, the positive x-axis, and the graph of $y = x^2$.

Solution

The area, $A = \int_0^1 x^2\,dx$

$$= \left.\frac{x^3}{3}\right|_0^1$$

$$= \frac{1^3}{3} - \frac{(0^3)}{3} = \frac{1}{3}.$$

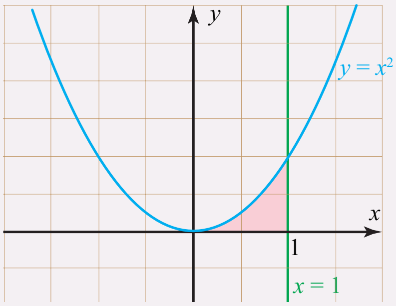


Figure 12.

Therefore, in the figure 12 the shaded region has an area $\frac{1}{3}$ square units.

Example 68

Find the area bounded by the graph of $f(x) = x^2 + 10$, the x-axis, and the lines $x = -3$ and $x = 7$.

Solution

$$A = \int_{-3}^{7} f(x)\,dx$$

$$= \int_{-3}^{7} (x^2 + 10)\,dx = \left.\frac{x^3}{3} + 10x\right|_{-3}^{7}$$

$$= \frac{7^3}{3} + 7(10) - \left(\frac{(-3)^3}{3} + 10(-3)\right)$$

$$= \frac{343}{3} + 70 + 9 + 30 = \frac{670}{3}.$$

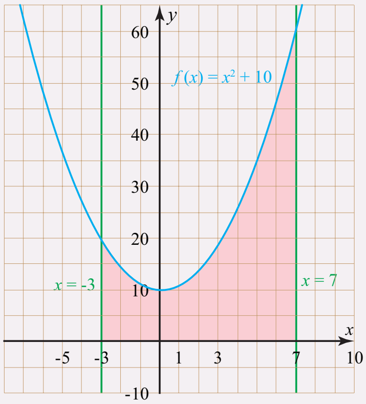


Figure 13.

Example 69

Find the area bounded by the x-axis between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ and the graph of $f(x) = \cos x$.

Solution

$$A = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\cos x)\, dx = \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= 1 - (-1) = 2.$$

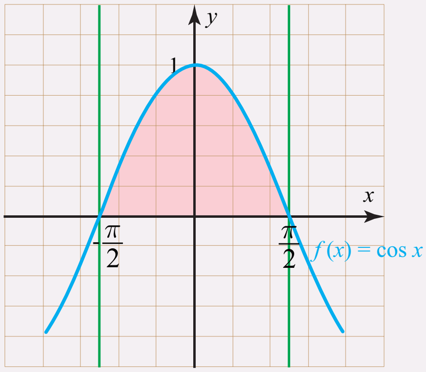


Figure 14.

Example 70

Find the area of the region bounded by the graph of $f(x) = 36 - x^2$ and the x-axis.

Solution

The first step to solve this problem is to draw the graph of f.

$$A = \int_{-6}^{6} (36 - x^2)\, dx$$

$$= 36x - \frac{x^3}{3} \Big|_{-6}^{6}$$

$$= 36(6) - \frac{6^3}{3} - \left(36(-6) - \frac{(-6)^3}{3}\right) = 288.$$

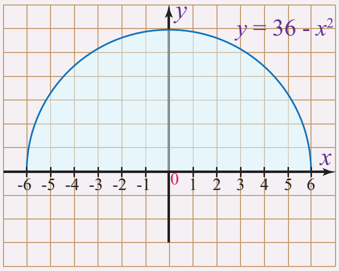


Figure 15.

Example 71

Find the area bounded by the x-axis at $x = 0$ and $x = \frac{\pi}{2}$ and the graph of $f(x) = \sin x$.

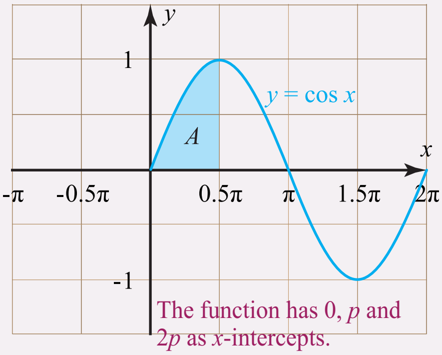


Figure 16.

Solution

$$A = \int_0^{\frac{\pi}{2}} \sin x \, dx = -\cos x \int_0^{\frac{\pi}{2}}$$

$$= -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$$

$$= -[-0 - 1] = 1.$$

Exercises

Find the area bounded by the graph of $f(x)$ and the x-axis between the given specific values of x.

1. $f(x) = x + 5$; $x = -10$ and $x = 3$
2. $f(x) = x^2$; $x = -3$ and $x = 4$
3. $f(x) = x^2 - 2x + 3$; $x = 0$ and $x = 4$
4. $f(x) = -x^2 + 4x$; $x = 01$ and $x = 3$
5. $f(x) = 9x - x^2$; $x = -2$ and $x = 10$
6. $f(x) = x^3 - x^2 - x + 1$; $x = -1$ and $x = 1$
7. $f(x) = 4x^2 - x^4$; $x = -2$ and $x = 2$
8. $f(x) = \sin x$; $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$

Key Terms

- Area under a curve
- Definite integral
- Derivative
- Differentiation
- Difference quotient
- Difference rule
- Distance formula
- Equation of a line
- Integration
- Indefinite integral
- Limits
- Mid - point formula
- Parallel lines
- Perpendicular lines
- Product rule
- Power rule
- Quotient rule
- Secant line
- Slope
- Sum rule
- Tangent line

SUMMARY

- **Distance formula**

  The distance of a point at $P(x_o, y_o)$ from the line $Ax + By + C = 0$ is

  $$d = \frac{|Ax_o + By_o + C|}{\sqrt{A^2 + B^2}}.$$

- **Midpoint formula**

  The midpoint of a line segment with end points at $A(x_1, y_1)$ and $B(x_2, y_2)$ is at

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

- **Slope formula**

  The slope of the line through the paint at $A(x_1, y_1)$ and $B(x_2, y_2)$ with $x_1 \neq x_2$ is

  $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **Equation of a line**

  (i) **Two point form**

  $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

  (ii) **Point slope form**

  $$y - y_1 = m(x - x_1).$$

  (iii) Slope intercept form

  $$y = mx + b.$$

  (iv) General form $Ax + By + C = 0$; either A, B or both different from zero.

- Parallel Lines

  Two non-vertical lines are parallel if and only if they have equal slopes.

  The equation of a non - vertical line passing through (x_o, y_o) and parallel to the line $Ax + By + C = 0$ is $A(x - x_o) + B(y - y_o) = 0$.

- **Perpendicular lines**

  Two non-vertical liens are perpendicular if an only if the product of their slopes is -1.

  The equation of the line passing through (x_o, y_o) and perpendicular to the line $Ax + By + C = 0$ is $A(y - y_o) - B(x - x_o) = 0$.

- **Limits**

  Let $f(x)$ be defined on a open interval about x_o, except possibly at x_o itself. If $f(x)$ gets closer and closer to a unique real number L for all x sufficiently close to x_o, then we say that f approaches the limit L as x approaches x_o, and we write $\lim_{x \to x_o} f(x) = L$.

- **Derivative at a point**

  (i) The expression $f(x) = \dfrac{f(x_o + h) - f(x_o)}{h}$ is said to be the difference quotient of f at x_o with incensement h.

  (ii) $\lim_{h \to 0} \dfrac{f(x_o + h) - f(x_o)}{h}$ exist, this limit is said to be the derivatives of f at x_o It is also said to be

  - The slope of $y = f(x)$ at $(x_o, f(x_o))$
  - The slope of the tangent line to the graph of $y = f(x)$ at $(x_o, f(x_o))$.
  - The rate of change of $f(x)$ with respect to x at $(x_o, f(x_o))$.

- **Notation**

  The derivative of $y = f(x)$ with respect x is denoted by $f'(x)$ or $\dfrac{dy}{dx}$.

- The equation of the tangent line

  The equation of the line tangent to the graph of $y = f(x)$ at $(x_o, f(x_o)$ is $y = f'(x_o)(x - x_o) + f(x_o)$.

- **The Indefinite integral or Anti derivative**

  If $F'(x) = f(x)$, then $F(x) = \int f(x)dx$

  $F(x)$ is said to be the indefinite integral of $f(x)$ or antiderivative of $f(x)$.

  - $\int$ is the integral sign
  - $f(x)$ is the integral
  - x is the variable of integration

- **The definite integral**

  A continuous function is integrable or has an integral on an interval $[a, b]$ if it is continuous on $[a, b]$.

- **Notation for definite Integral**

  - $\int_a^b f(x)dx$ is symbol for definite integral.

- a is said to be the lower limit of integration.
- b is said to be the upper limit of integration.

- **The Fundamental Theorem of Calculus**

  If f is continuous on $[a, b]$ and F' is the derivative of f on $[a, b]$, then

  $\int_a^b f(x)dx = F(b) - F(a).$

- **Area under a curve**

  Let f be continuous on $[a, b]$.

  If $f(x) \geq 0$ on $[a, b]$, then the area of the region bounded by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is $\int_a^b f(x)dx$.

EXERCISES

Find the value of k if the distance between $(3, k)$ and $(4, 7)$ is 21.

1. Find the slope of the line through the points (a, b) and (b, a)
2. Are the points (2, 3), (4, 7) and (6, 9) collinear?
3. Write the equation of a line
   (a) Passing through (2, 5) and (–5, 2)
   (a) Slope 3 and passing through (–5, 4)
   (a) x-intercept –2 and y-intercept –3
   (b) Slope 5 and passing through the origin.
4. Find the equation of the line which is perpendicular to $3x - 5y + 13 = 0$ and passing through (–2, –5).
5. Find the x and y-intercepts of each of the following lines.
   (a) $8x + 5y = 1$ (b) $y - 2x + 7 = 0$
   (c) $\frac{x}{a} + \frac{y}{b} = ab \neq 0$ and a and b are constants
   (d) $x = 5y + 11$
6. Find the distance of each point from the given line
   (a) $(3, -4)$ from $2x + y = 1$ (b) $(5, 1)$ from $y = 2x - 11$
   (c) $(0, 0)$ from $ax + by + c = 0$; $ab \neq 0$
7. Find the distance between the following lines
   (a) $y = 2x + 1$ and $y = 2x - 5$ (b) $2x + y = 3$ and $2x + y = 11$

8. If the lines $y = ax - 3$ and $y = \frac{3}{a}x$ are parallel, find the set of values of a.

9. Evaluate the following limits.

   (a) $\lim_{x\to 1}(3x-2)$

   (b) $\lim_{x\to -2}(x^2-5x+4)$

   (c) $\lim_{x\to -5}\frac{x^2-25}{x+5}$

   (d) $\lim_{x\to 2}\frac{x^2-3x-10}{x^2-4}$.

10. Find the difference quotient of

   (a) $f(x) = 5x + 2$

   (b) $f(x) = 3 - 7x$

   (c) $f(x) = x^2 - 10x + 4$

   (d) $f(x) = x^3 - x^2 + 2$.

11. Find the slope of the graph of

   (a) $f(x) = x^2 - x^3 + 10$ at $x = 2$

   (b) $f(x) = 2 - 4x - 5x^2$ at $x = -3$.

   (c) $f(x) = \sin x$ at $(0, 0)$

12. Differentiate the following functions with respect to x.

   (a) $f(x) = 4x^5$

   (b) $f(x) = \sqrt{2}x^7$

   (c) $f(x) = x - \sin x + 4\cos x + 12$

   (d) $f(x) = x^3 - 4x^2 + 2\cos x + 5$

   (e) $f(x) = \frac{-3x^2+4x^3}{5x^2}$.

13. Integrate the following functions with respect to x.

   (a) $f(x) = 6x - 1$

   (b) $f(x) = 3x^2 - 4x^2 + 5x + 1$

   (c) $f(x) = 2x^3 - 4x^2 - 5x + 1$

   (d) $f(x)\ \sqrt{2}\cos x - x^2 - 3$

   (e) $f(x) = 3 + \sin x + 2\cos x$.

14. Evaluate the following definite integers

   (a) $\int_0^1 (6x^2+5x)dx$

   (b) $\int_{-1}^2 (x^3+x^2-x+1)dx$

   (c) $\int_0^{\pi} x^3 dx$

   (d) $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (5-\sin x+\cos x)dx$.

15. Find the area of the region under the graph of

   (a) $f(x) = x^2$ between $x = -1$ and $x = 3$

   (b) $f(x) = 9x - x^2$

   (c) $f(x) = \sin x$ between $x = \frac{\pi}{4}$ and $x = \frac{3}{4}\pi$

   (d) $f(x) = 6 + x - x^2$.

Choose the best answer

1. The sum of the numbers between 100 and 1000 which is divisible by 9 will be

   (a) 55350 (c) 97015

   (b) 57228 (d) 62140

2. Let and be roots of $x^2 - 3x + p = 0$ and let c and d be the roots of $x^2 - 12x + q = 0$, where a, b, c, d form an increasing G.P. Then the ratio of $(q + p) : (q - p)$ is equal to

   (a) 8 : 7 (b) 11 : 1 (c) 17 : 15

3. If the product of three consecutive terms of G.P. is 216 and the sum of product of pair-wise is 156, then the numbers will be

   (a) 1, 3, 9 (c) 3, 9, 27

   (b) 2, 6, 18 (d) 2, 4, 8

4. If the p^{th} term of an A.P. be q and q^{th} term be p, then its r^{th} term will be

   (a) $p + q + r$ (c) $p + r - q$

   (b) $p + q - r$ (d) $p - q - r$

5. The sum of numbers from 250 to 1000 which are divisible by 3 is

   (a) 135657 (c) 161575

   (b) 136557 (d) 156375

6. The ratio of sum of m and n terms of an A.P. is $m^2 : n^2$, then the ratio of m^{th} and n^{th} term will be

   (a) $m - 1 / n - 1$ (c) $2m - 1 / 2n - 1$

   (b) $n - 1 / m - 1$ (d) $2n - 1 / 2m - 1$

7. The sum of all natural numbers between 1 and 100 which are multiples of 3 is

   (a) 1680 (c) 1681

   (b) 1683 (d) 1682

8. The number of terms of the A.P. 3,7,11,15…to be taken so that the sum is 406 is

   (a) 5 (b) 10 (c) 12 (d) 14

9. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

   (a) 3000 (c) 4050

   (b) 3050 (d) None of these

10. There are 15 terms in an arithmetic progression. Its first term is 5 and their sum is 390. The middle term is

(a) 23 (b) 26 (c) 29 (d) 32

11. There are 15 terms in an arithmetic progression. Its first term is 5 and their sum is 390. The middle term is

(a) 23 (b) 26 (c) 29 (d) 32

12. 7^{th} term of an A.P. is 40, then the sum of first 13 terms is

(a) 53 (b) 520 (c) 1040 (d) 2080

13. If the sides of a right angled triangle are in A.P., then the sides are proportional to

(a) 1: 2: 3 (b) 2: 3: 4 (c) 3: 4: 5 (d) 4: 5: 6

14. If the ratio of the sum of first three terms and the sum of first six terms of a G.P. be 125 : 152, then the common ratio r is

(a) 3/5 (b) 5/3 (c) 2/3 (d) 3/2

15. If the first term of an A.P. be 10, last term is 50 and the sum of all the terms is 300, then the number of terms are

(a) 5 (b) 8 (c) 10 (d) 15

16. If sum of infinite terms of a G.P. is 3 and sum of squares of its terms is 3, then its first term and common ratio are

(a) $\frac{3}{2},\frac{1}{2}$ (c) $\frac{3}{2},2$

(b) $1,\frac{1}{2}$ (d) None of these

17. The average (arithmetic mean) of A and B is 15, and the average of C and D is 17. What is the average of A, B, C and D?

(a) 15 (b) 16 (c) 17 (d) 18

18. If $a = 64 + 64 + 64 + \ldots + 64$ (100 terms) and $b = 8 + 8 + 8 + \ldots + 8$ (100 terms), then $\frac{a}{b}$ is __?

(a) 24 (b) 16 (c) 8 (d) 64

19. The sum of the first 100 odd natural numbers is _____?

(a) 5050 (c) 5000

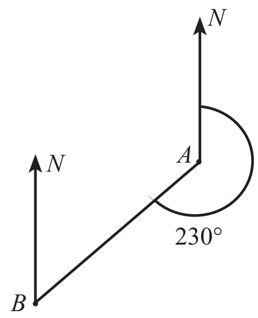
(b) 10000 (d) 000

20. How many two digit numbers are there divisible by 5?

(a) 15 (b) 20 (c) 19 (d) 18

Choose the best answer

1. By looking at the figure which one is the bearing of A from B?
   (a) 130° (c) 230°
   (b) 050° (d) 290°



2. If the bearing of D from E is 223°, what is the bearing of E from D?
   (a) 043° (c) 088°
   (b) 137° (d) 092°

Choose the best answer

1. When copying line segment AB using a straight edge and a compass, the compass should be used to:
   (a) Draw an arc above point A
   (b) Measure the length of segment AB
   (c) Draw an arc between point A and point B
   (d) Measure half the length of line segment AB
2. Suppose we wish to construct $\angle EFG$ congruent to $\angle DBC$ using a compass and straight edge. Which step would be correct to do first?
   (a) Place the compass point at B
   (b) Place the compass point at C
   (c) Place the straightedge along A and C
   (d) Place the straightedge along C and D

Choose the best answer

1. Which of the following cannot be obtained easily through observation from the frequency distribution table of ungrounded quantitative data?
   (a) Largest value of the data set.
   (b) Mode of the data set.

(c) Smallest value of the data set.

(d) Mean.

2. Given a test score of 10 students as: 13, 14, 12, 11, 15, 14, 13, 11, 12, 15 out of 15, the mean score is ______________.

(a) 14 (b) 13 (c) 12 (d) 12.5

3. Which of the following is true?

(a) Mode must be the value in the data.

(b) Mode is used for both qualitative and quantitative data.

(c) Mode is uniquely determined.

(d) Mode is the one which occurs exactly at half way in the data set.

4. Which of the following is the property of mean?

(a) There can be a double mean.

(b) It cannot be the value outside the data set.

(c) To calculate it we may not need all the data values.

(d) It is affected by extreme values.

5. Which of the following is the use of measure of dispersions?

(a) They indicate the value of the quartiles.

(b) They locate the center of the data set.

(c) They indicate the popular value in the data set.

(d) They measure how the values in the data set vary.

6. The average blood pressure of a person taken for test six times is 74 cc. When it is taken for the 7^{th} time the average blood pressure of the person becomes 75 cc. How much is the blood pressure of the person in the 7^{th} test?

(a) 76 (b) 78 (c) 81 (d) 80

7. If x is the median of the 7 numbers: 10, 18, 6, 12, 20, x, 14 then which of the following could be the value of x?

(a) 9 (b) 11 (c) 13 (d) 15

For questions 8 and 9 use the following data.

25, 10, 10, 18, 13, 11, 30, 22, 28, 28, 17, 11, 18, 24, 19, 16, 13, 25, 32, 27

8. The first and the second quartiles Q_1 and Q_2 are:

(a) $Q_1 = 13$ and $Q_2 = 18$ (c) $Q_1 = 10$ and $Q_2 = 18$

(b) $Q_1 = 10$ and $Q_2 = 16$ (d) $Q_1 = 18$ and $Q_2 = 25$

9. The quartile deviation of the data is ____________

   (a) 5 (b) 6 (c) 12 (d) 8

10. The standard deviation of the data is ____________

    (a) $\sqrt{43.5}$ (c) $\sqrt{73.5}$

    (b) $\sqrt{33.5}$ (d) $\sqrt{63.5}$

Choose the best answer

1. The equation of the line L is given by $2x - 3y + 7 = 0$. One of the following is the equation of a line parallel to L.

   (a) $3x + y + 5 = 0$ (c) $3x - 2y - 5 = 0$

   (b) $6x - 9y + 5 = 0$ (d) $6x + 2y - 5 = 0$

2. The solution set of the system: $\dfrac{6x - 4y = 20}{6x + 6y = 15}$ is ____________.

   (a) $\{(-1/2, 3)\}$ (c) $\{(3, -1/2)\}$

   (b) $\{(1/2, -3)\}$ (d) $\{(-1/2, 2)\}$

Choose the best answer

1. The curved surface area of a right circular cone of height 15 cm and base diameter 16 cm is

   (a) 60π cm^2 (c) 120π cm^2

   (b) 68π cm^2 (d) 136π cm^2

2. If two solid hemispheres of same base radius r units are joined together along their bases, then curved surface area of this new solid is

   (a) $4\pi r^2$ sq. units (c) $3\pi r^2$ sq. units

   (b) $6\pi r^2$ sq. units (d) $8\pi r^2$ sq. units

3. The height of a right circular cone whose radius is 5 cm and slant height is 13 cm will be

   (a) 12 cm (c) 13 cm

   (b) 10 cm (d) 5 cm

4. If the radius of the base of a right circular cylinder is halved keeping the same height, then the ratio of the volume of the cylinder thus obtained to the volume of original cylinder is

(a) 1:2 (b) 1:4 (c) 1:6 (d) 1:8

5. The total surface area of a cylinder whose radius is 1/3 of its height is

(a) $\frac{9\pi h^2}{8}$ sq.units (c) $\frac{8\pi h^2}{9}$ sq.units

(b) $24\pi h^2$ sq.units (d) $\frac{56\pi h^2}{9}$ sq.units

6. In a hollow cylinder, the sum of the external and internal radii is 14 cm and the width is 4 cm. If its height is 20 cm, the volume of the material in it is

(a) 5600π cm^3 (c) 56π cm^3

(b) 11200π cm^3 (d) 3600π cm^3

7. If the radius of the base of a cone is tripled and the height is doubled then the volume is

(a) made 6 times (c) made 12 times

(b) made 18 times (d) unchanged

8. The total surface area of a hemi-sphere is how much times the square of its radius.

(a) π (b) 4π (c) 3π (d) 2π

9. A solid sphere of radius x cm is melted and cast into a shape of a solid cone of same radius. The height of the cone is

(a) $3x$ cm (b) x cm (c) $4x$ cm (d) $2x$ cm

10. A frustum of a right circular cone is of height 16cm with radii of its ends as 8cm and 20 cm. Then, the volume of the frustum is

(a) 3328π cm^3 (c) 3240π cm^3

(b) 3228π cm^3 (d) 3340π cm^3

11. The volume (in cm^3) of the greatest sphere that can be cut off from a cylindrical log of wood of base radius 1 cm and height 5 cm is

(a) $4/3\ \pi$ (b) $10/3\ \pi$ (c) $5\ \pi$ (d) $20/3\ \pi$

12. The ratio of the volumes of a cylinder, a cone and a sphere, if each has the same diameter and same height is

(a) 1:2:3 (b) 2:1:3 (c) 1:3:2 (d) 3:1:2

13. If the diameter of a sphere is 12 cm, then its surface area is ____________.

 (a) 200π cm^2 (c) 140π cm^2

 (b) 576π cm^2 (d) 144π cm^2

14. The volume of circular cylinder with base radius 3 cm and height 4 cm is ____________.

 (a) 72π cm^3 (c) 108π cm^3

 (b) 54π cm^3 (d) 36π cm^3

15. A regular pyramid has square base whose side is 8 cm long. If the length of the lateral edge is 12 cm, then its total area is ________________.

 (a) $64(1+2\sqrt{2})$ cm^2 (c) $128\sqrt{2}$ cm^2

 (b) $64\sqrt{5}$ cm^2 (d) $64\left(1+\sqrt{5}\right)$ cm^2

16. If the radius of a sphere is 10 cm, then its surface area is ________________.

 (a) 200π cm^2 (c) 100π cm^2

 (b) 400π cm^2 (d) $\frac{400}{3}\pi$ cm^2

Choose the best answer

1. The converse of the statement: If patients get good treatment, then they feel comfortable.
   (a) If patients feel comfortable, then they get good treatment.
   (b) If patients do not feel comfortable, then they get good treatment.
   (c) If patients feel comfortable, then they do not get good treatment.
   (d) If patients do not feel comfortable, then they do not get good treatment.
2. An equivalent statement to the statement given in question number (1) is,
   (a) If patients feel comfortable, then they get good treatment.
   (b) If patients do not feel comfortable, then they get good treatment.
   (c) If patients feel comfortable, then they do not get good treatment.
   (d) If patients do not feel comfortable, then they do not get good treatment.

Choose the best answer

1. Which of the following statements is **not** true?
   (a) 40% of 25 LRD is 10 LRD.
   (b) 2.5% of 80 meters is 2 meters.
   (c) 12 % of 75 gms is 8.5 gms.
   (d) 30% of 18 litters is 5.4 litters.
2. If $12\frac{1}{2}\%$ of a certain number is 50, what is the number?
   (a) 400 (b) 420 (c) 360 (d) 320
3. A department store gives 15% discount on an article priced at 2000 LRD. What is the amount of discount?
   (a) 180 LRD (c) 270 LRD
   (b) 300 LRD (d) 225 LRD
4. A factory has 1200 workers of which 720 are men and the rest are women. What percent of the workers are women?
   (a) 40% (b) 60% (c) 33% (d) 67%
5. The price of a shirt at a shop is 120 LRD plus 15% VAT. What would be the total cost of the shirt?
   (a) 240 LRD (c) 135 LRD
   (b) 138 LRD (d) 142 LRD
6. A box contains 75 red, blue and green balls. If 24% are red and 17 are blue, what fraction of the balls is green?
   (a) $\frac{7}{15}$ (b) $\frac{8}{15}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$
7. A trader bought a TV set for 2000 LRD and sold it at a loss of 5 %, what was the selling price?
   (a) 1900 LRD (c) 1850 LRD
   (b) 1890 LRD (d) 1800 LRD
8. Ahmed's salary last year was 2500 LRD. This year he got 15% increment. What is his salary at present?
   (a) 2800 LRD (c) 2875 LRD
   (b) 2900 LRD (d) 2975 LRD

9. A merchant gained 15% by selling an article for 230 LRD. What was the cost price of the article?
   (a) 250 LRD (c) 200 LRD
   (b) 240 LRD (d) 180 LRD
10. How long will it take 300 LRD to double itself if it is invested at the rate of 5% simple interest per annum?
   (a) 10 years (c) 20 years
   (b) 15 years (d) 12 years

Choose the best answer

1. Two triangles are similar. The length of a side of one triangle is 3 times that of the length of the corresponding side of the other. What is the ratio of the perimeter of the larger to the perimeter of the smaller triangle?
   (a) 9:1 (b) 3:1 (c) 6:3 (d) 6:1
2. Two triangles ate similar. The ratio of the length of a side of one triangle to a corresponding side of the other is 2:3. If the perimeter of the larger triangle is 12 cm, what is the perimeter of the smaller triangle?
   (a) 8 cm (b) 9 cm (c) 10 cm (d) 6 cm
3. Two triangles are similar. The length of a side of one triangle is 3 times that of the length of the corresponding side of the other. What is the ratio of the area of the larger to the area of the smaller triangle?
   (a) 1:4 (b) 9:1 (c) 3:1 (d) 6:1
4. *ABCD* and *PQRS* are rectangles such that $ABCD \sim PQRS$. If AB = 3cm, PQ = 6cm and area of *ABCD* is 15cm^2. What is the area of *PQRS*?
   (a) 30 cm^2 (b) 60 cm^2 (c) 45 cm^2 (d) 90 cm^2
5. Which of the following plane figures are not necessarily similar?
   (a) Two parallelograms (c) Two circles
   (b) Two *n* sided regular polygons (d) Two Equilateral Triangles
6. ΔABC and ΔDEF are right – angled triangles with $m(\angle B) = m(\angle E) = 90°$. If $\Delta ABC \sim \Delta DEF$ and AB = 20cm, BC = 15cm and DE = 6cm, then what is the area of ΔDEF?
   (a) 13.5cm^2 (b) 24cm^2 (c) 45cm^2 (d) 150cm^2

7. Two triangles are similar. The length of a side of the first triangles is 1.5 times as long as the length of the corresponding side of the other, then which of the following is not true?
   (a) The perimeter of the larger triangle will be 1.5cm, if the perimeter of the smaller one is 10cm.
   (b) If the area of the smaller triangle is $6cm^2$, then the area of the larger triangle will be $9cm^2$.
   (c) If a side of the larger triangle is 9 cm long, then the corresponding side of the smaller triangle will be 6 cm long.
   (d) If the sum of the lengths of the two sides of the larger triangle is 9 cm^2, then the sum of the lengths of the two corresponding sides of the smaller triangle is 6 cms.
8. If two triangles are similar with ratio of area $\frac{1}{9}$, then what is the ratio of their corresponding sides?
   (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{81}$ (d) $\frac{2}{3}$
9. ΔABC is similar to ΔDEF, if $AB = 6$cm and $DE = 2$ cm, then what is the ratio of area of ΔABC to ΔDEF?
   (a) 3:1 (b) 9:1 (c) 4:1 (d) 2:1
10. $ABCD$ and $PQRS$ are rectangles such that $ABCD \sim PQRS$. If $AB = 3$cm $PQ = 6$cm and area $ABCD$ is $15cm^2$, then what is the area of $PQRS$?
   (a) $30cm^2$ (b) $60cm^2$ (c) $45cm^2$ (d) $90cm^2$

Choose the best answer

1. If the attitude of an equilateral triangle is a cm, then the area of the triangle is:
   (a) $\frac{a^2\sqrt{3}}{2}cm^2$ (c) $\frac{a^2\sqrt{3}}{3}cm^2$
   (b) $\frac{a^2\sqrt{3}}{4}cm^2$ (d) $\frac{3a^2\sqrt{3}}{4}cm^2$
2. Which of the following is negative?
   (a) cos (–89°) (c) tan(–120°)
   (b) sin (–89°) (d) sin (120°)

3. For an acute angle A if $\sin A = \frac{5}{13}$, then $\cos A$ is equal to

(a) $\frac{12}{13}$ (b) $\frac{5}{12}$ (c) $\frac{-12}{13}$ (d) $\frac{-5}{12}$

4. For an obtuse angle A if $\cos A = \frac{-11}{61}$, then $\sin A$ is equal to

(a) $\frac{11}{60}$ (b) $\frac{-11}{60}$ (c) $\frac{60}{61}$ (d) $\frac{-60}{61}$

5. Which of the following is not equal to $\sin 60°$?

(a) $2 \sin 30° \cos 30°$

(b) $\frac{1}{\sqrt{2}}\sqrt{1-\cos 120°}$

(c) $\sin 20° \cos 40° + \cos 20° \sin 40°$

(d) $\sqrt{\cos^2 60° - \cos 120°}$

6. If $\sin \theta = 0.6$, then $\cos^2 \theta$ is

(a) 0.8 (b) 0.64 (c) 0.36 (d) 0.4

7. Which of the following identities is not true for all values of t?

(a) $\cos(\pi - t) = -\cos t$

(b) $\cos\left(\left(\frac{\pi}{2}\right) - t\right) = \sin t$

(c) $\sin\left(\left(\frac{\pi}{2}\right) - t\right) = \cos t$

(d) $\sin(\pi - t) = -\sin t$

8. If the measure of an angle is $583°$ in clockwise direction, then in which quadrant does its terminal side fall?

(a) III (b) IV (c) I (d) II

9. Which of the following is not equal to $\cos 72°$?

(a) $\cos(-648)°$

(b) $\cos 792°$

(c) $\sin 180°$

(d) $\cos 432°$

10. From the following trigonometric values which one is equal to $\sin(420°)$?

(a) $\sin(-300°)$

(b) $\sin(-780°)$

(c) $\sin(30°)$

(d) $\sin(-60°)$

Choose the best answer

1. Which one of the following is the fractional form of the decimal number 4.8121212…

   (a) $\frac{4764}{990}$ (b) $\frac{4764}{999}$ (c) $\frac{4764}{900}$ (d) $\frac{44}{9}$

2. To rationalize the denominator of $\frac{7}{3+\sqrt{2}}$, which one of the following expression is used as the rationalizing factor.

   (a) $\frac{3+\sqrt{2}}{3+\sqrt{2}}$ (c) $\frac{3-\sqrt{2}}{3-\sqrt{2}}$

   (b) $\frac{\sqrt{2}-3}{\sqrt{2}-3}$ (d) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

3. The simplified form of the expression $\frac{2\sqrt{72}}{3}-\frac{3\sqrt{128}}{4}+5\sqrt{\frac{1}{2}}$ is equal to

   (a) $\sqrt{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $18\sqrt{2}$ (d) 1

4. If x is a rational number and y is an irrational number, then which of the following is necessarily true?

   (a) xy is an irrational number. (c) $\frac{x}{y}$ is an irrational number

   (b) $\frac{x}{y}$ is a real number (d) x , y is an irrational number

5. The number $\sqrt{0.0625}$ does not belong to one of the following intervals, which one is that?

   (a) (0.2, 0.3) (c) $\left(\frac{1}{5},\frac{1}{3}\right)$

   (b) (0. 225, 0.55) (d) (0. 25, 0.275)

6. $\frac{3\sqrt{24}-2\sqrt{18}}{-\sqrt{2}}$ is equal to

   (a) $6\left(1-\sqrt{3}\right)$ (c) $6\sqrt{2}\left(\sqrt{3}-1\right)$

   (b) $6\left(\sqrt{3}-1\right)$ (d) $6\sqrt{2}\left(1-\sqrt{3}\right)$

7. If a, b, c are non–zero real numbers such $a > b > c$, then which of the following is always true?

(a) $ab > bc$ (c) $a - b > b - c$

(b) $a + b > b + c$ (d) $\frac{a}{b} > \frac{b}{c}$

8. When the expression $\left(\frac{5}{3} - \frac{7}{4}\right) + \sqrt{\frac{1}{16} + \frac{1}{9}}$ is simplified it gives:

(a) $\frac{2}{3}$ (b) $\frac{7}{12}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

9. When simplified $\frac{\sqrt{5 - 2\sqrt{6}} \times \sqrt{5 + 2\sqrt{6}}}{\sqrt[3]{125}}$ give:

(a) $\sqrt{\frac{13}{5}}$ (b) $-\frac{\sqrt{13}}{5}$ (c) $\frac{-1}{5}$ (d) $\frac{1}{5}$

10. For real numbers x and $\frac{1}{x}$, which one of the following statements is not true about them?

(a) If $0 < x < 1$, then $\frac{1}{x} < 1$ (c) If $-1 < x < 0$, then $\frac{1}{x} < -1$

(b) If $x < -1$, then $-1 < \frac{1}{x} < 0$ (d) If $x > 1$, then $0 < \frac{1}{x} < 1$.

Choose the best answer

1. If $A = \{-1, \varnothing, 1\}$ and $B = \{-2, \varnothing, 2\}$, then which of the following is not true?

(a) $A \cap B = \{\varnothing\}$ (c) $B \backslash A = \{-2, 2\}$

(b) $A \backslash B = \{-1, 1\}$ (d) $A \cap B = \varnothing$

2. If $A = \{\varnothing\}$, then which of the following is not true?

(a) $A \subseteq \{\{\varnothing\}\}$ (c) $\varnothing \subseteq A$

(b) $\varnothing \subseteq \{A\}$ (d) $A \subseteq \{\{\varnothing\}\}$

3. If $B \subseteq A$, then which of the following is not true?

(a) $A \cup B = A$ (c) $A \backslash B = \varnothing$

(b) $A \cap B = B$ (d) $A' \subseteq B'$

4. Which of the following is not a subset of $\{\varnothing, \{0\}\}$?

(a) $\{\varnothing\}$ (b) $\{\{0\}\}$ (c) $\varnothing$ (d) $\{0\}$

5. Let $\mathbb{N}$ be the set of all natural numbers and $S=\{x\in\mathbb{N}|x$ is a prime number and $x<5\}$. Which one of the following is the list of all the proper subsets of set S?
   (a) $\varnothing,\{2\},\{3\},\{2,3\}$
   (b) $\varnothing,\{1\}\{2\},\{3\},\{1,2\},\{1,3\}\{2,3\}$
   (c) $\varnothing,\{2\},\{3\},\{2,3\}$
   (d) $\varnothing,\{2\},\{3\}$
6. If set $\mathbb{N}=\{x|-2<x<2, \text{where } x \text{ is an integer}\}$, then which one of the following sets is equivalent to set M?
   (a) $\{x\,|\,x<3, \text{where } x \text{ is a natural number}\}$
   (b) $\{x\,|\,1<x<5, \text{where } x \text{ is an integer}\}$
   (c) $\{x\,|-1<x<1, \text{where } x \text{ is an integer}\}$
   (d) $\{x\,|-2<x<2, \text{where } x \text{ is a real number}\}$
7. Let $H=\{\varnothing,\{0,1\}\}$. From the following sets, which one is the power set of set?
   (a) $\{\varnothing,\{0\},\{1\},\{0,1\}\}$
   (b) $\{\varnothing,\{\varnothing\}, \{\{0,1\}\},\{\varnothing,\{0,1\}\}\}$
   (c) $\{\{\varnothing\},\{\{0,1\}\},\{\varnothing,\{0,1\}\}\}$
   (d) $\{\varnothing, \{\{0,1\}\},\{\varnothing,\{0,1\}\}\}$
8. For two finite sets A and B where $A\subset B$ with $n(A)=p$ and $n(B)=q$ such that $p<q$, then which one of the following relationships is true?
   (a) $n(A\cup B)=p+q$
   (b) $n(A\cup B)<q$
   (c) $n(A\cup B)=p$
   (d) $n(A\cup B)=q$
9. Let $S=\{x\in\mathbb{R}:-3<x<4\}$. Which one of the following statements is true about the set S?
   (a) The number of subsets of set S is 64.
   (b) Only one subset of S has exactly 6 elements.
   (c) $\frac{1}{2}$ is an element of set S.
   (d) The number of elements of set S is 6.

10. Let the universal set U be the set of positive integers and let set $S=\{x\in U|x\text{ is a prime number and }x^2\leq 30\}$. Which one of the following sets is equal to set S?

(a) $\{2,3,5,7,11,13,17,19,23,29\}$ (c) $\{4,9,25\}$

(b) $\{4,9,16,25\}$ (d) $\{2,3,5\}$

11. Among the 250 regular customers of a shop, 149 of them regularly buy product A, 131 customers regularly buy product B. If 20 of the customers buy neither of the products, then what is the number of customers that regularly buy only product A?

(a) 99 (b) 70 (c) 50 (d) 83

12. Which one of the following sets is a finite set?

(a) The set of points on the number line between -1 and 1.

(b) The set of positive composite numbers less than 1000.

(c) The set of irrational numbers between 0 and 5.

(d) The set of positive prime numbers.

13. If the universal set is the set of integers, then which one of the following pairs of sets contains sets that are **NOT** equivalent?

(a) $\{x:3x+5=0\}$ and $\{x:2x-7=0\}$

(b) $\{x:(x+2)(3x-2)=0\}$ and $\{x:x-2=0\}$

(c) $\{x:2x+3=2x-3\}$ and $\{x:3x=2\}$

(d) $\{x:(x-2)^2=9\}$ and $\{x:3x-1=5\}$

14. In a class of 52 students; 34 students passed in Mathematics and 28 passed in English. If the number of students who failed in both subjects is 9, then what is the number of students who passed in both subjects?

(a) 9 (b) 6 (c) 10 (d) 19

Choose the best answer

1. If $f(x)=3x-3$ and $g(x)=2-3x$, then $(2f+3g)(x)$ is equal to

(a) $-3x$ (c) $-6x+5$

(b) $-9x^2+15x-6$ (d) $12x+6$

2. If a function is defined by $f(x) = -x^2 + 2x + 1$, which of the following is the range of this function?

(a) $\{y \in \mathbb{R} \mid y \geq 1\}$ (c) $\{y \in \mathbb{R} \mid y \geq 2\}$

(b) $\{y \in \mathbb{R} \mid y \leq 1\}$ (d) $\{y \in \mathbb{R} \mid y \leq 2\}$

3. If $p(x) = \sqrt{x} + 2$ and $q(x) = \frac{1}{\sqrt[3]{x}}$, then which one of the following is true?

(a) $(qp)(1) = 3$ (c) $(pq)(-1) = -3$

(b) $(pq)(4) = 2$ (d) $(pq)(0) = 2$

4. Let a function g be given as $g(x) = (1-a)x - b + 2$, where a and b are positive real numbers. Which of the following is true about the graph of g?

(a) The x-intercept is $\frac{b-2}{a-1}$ (c) It is increasing for $a < 1$

(b) The y- intercept is 2. (d) Its slope is $-a$

5. If a function is given by $p(x) = (1-2x)^2 + 4$, then what is the range of ?

(a) $(-\infty, \infty)$ (c) $[4, \infty)$

(b) $[8, \infty)$ (d) $(-\infty, -4]$

6. Let $f(x) = \sqrt{3-x}$ and $g(x) = x^2 + 2x + 1$, then which one of the following statement is correct?

(a) The domain of $\frac{g}{f}$ is $(3, \infty)$.

(b) The domain of $f + g$ is the set of all real numbers

(c) The domain of $\frac{f}{g}$ is $(-\infty, 3]$.

(d) The domain of fg is $(-\infty, 3]$.

7. The ratio of the age of a boy to the age of his father is 2:5. If the sum of their age is 70 years, how old is the father?

(a) 56 years (b) 50 years (c) 42 years (d) 40 years

8. A woman drives 120 km in 2 hours and a man drives 22.5 km in 30 minutes. What is the ratio of the woman's speed to that of the man's speed?

(a) 3:2 (b) 4:5 (c) 4:3 (d) 3:4

9. An alloy is made of copper, zinc and tin in the ratio 3:4:5, respectively. What is the amount of zinc in a 96 kg of the alloy?

(a) 32 kg (b) 48 kg (c) 40 kg (d) 24 kg

10. Suppose y varies inversely with x. If $y=9$, when x = 2, then find y when $x=3$.

(a) 3 (b) 6 (c) 12 (d) 18

Choose the best answer

1. Which one of the following systems of equations has infinitely many solutions?

(a) $\begin{cases} -2x+y=4 \\ 2x+y=4 \end{cases}$

(b) $\begin{cases} \frac{1}{2}x-2y=5 \\ x+4y=7 \end{cases}$

(c) $\begin{cases} \frac{2}{3}x+2y=5 \\ 2x+6y=2 \end{cases}$

(d) $\begin{cases} \frac{3}{2}x-2y=1 \\ 3x-4y=2 \end{cases}$

2. A business woman had invested 30,000 LRD on two business sectors. She had invested part of the money (x LRD) on one sector at 10% return and the rest of the money (y LRD) on the other sector at 5% return. If she received a total of 2300 LRD in return on the total investment, then which one of the following systems of linear equations describes this situation?

(a) $\begin{cases} x+y=32,300 \\ 2x+y=64,600 \end{cases}$

(b) $\begin{cases} x+y=32,300 \\ 2x+y=6,900 \end{cases}$

(c) $\begin{cases} x+y=30,000 \\ 10x+5y=2,300 \end{cases}$

(d) $\begin{cases} x+y=30,000 \\ 2x+y=46,000 \end{cases}$

3. From the following sets which one is the solution set of $2x^2+4x+1=0$?

(a) $\left\{-1-\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}\right\}$

(b) $\left\{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\}$

(c) $\{-1-\sqrt{2},-1+\sqrt{2}\}$ (d) $\varnothing$

4. Which one of the following system of linear equations has infinitely many solutions?

(a) $\begin{cases} x+y=6 \\ 2x-5y=0 \end{cases}$ (c) $\begin{cases} x-2y=5 \\ -3x+6y=1 \end{cases}$

(b) $\begin{cases} 3x+y=-1 \\ -x-\frac{1}{3}y=\frac{1}{3} \end{cases}$ (d) $\begin{cases} x+3y=2 \\ 2x-4y=1 \end{cases}$

5. Suppose the quadratic equation $ax^2+x-c=0$ has two real roots r_1 and r_2. If $r_1+r_2=r_1\times r_2=-\frac{1}{2}$, then c is equal to

(a) 2 (b) −2 (c) −1 (d) 1

6. What is the solution set of the equation $(x+8)(x-3)=3x$?

(a) $\{-3,2\}$ (b) $\{-8,3\}$ (c) $\{-6,4\}$ (d) $\{-4,6\}$

7. A rectangular plot of land has area 180 m^2. If the width is increased by 8 m and the length is reduced by 6 m , the area will still remain to be 180 m^2. What is the present width?

(a) 9 m (b) 20 m (c) 1 2m (d) 15 m

8. The difference between two numbers is 12. If 2 is added to seven times the smaller number, the result is the same as if 2 is subtracted from three times the larger number. What are the values of the smaller and the larger number respectively?

(a) 10 and 22 (b) 9 and 21 (c) 4 and 16 (d) 8 and 20

9. What is the product of roots of $5x^2-7x-6=0$?

(a) −3 (b) $\frac{-6}{5}$ (c) $\frac{-7}{5}$ (d) $\frac{7}{5}$

10. What is the solution set of the following system of equation $\begin{cases} 3x-0.5y=6 \\ -2x+y=4+2y \end{cases}$

(a) $\{(1,-6)\}$ (c) $\{(-6,1)\}$

(b) $\{(2, 0)\}$ (d) $\{\ \}$

Choose the best answer

1. Let $\vec{u} = \sqrt{2}\text{i} - \sqrt{3}\text{j}$ and $\overleftarrow{w} = \sqrt{3}\text{i} + \sqrt{2}\text{j}$ be vectors. Which of the following is the magnitude of the vector $\sqrt{2}\,\vec{u} + \sqrt{3}\,\vec{w}$
   (a) $\sqrt{5}$ (b) 5 (c) 10 (d) 15

2. Let $v_1 = \begin{pmatrix} \sqrt{5} \\ -\sqrt{2} \end{pmatrix}$ and $v_2 \begin{pmatrix} -2\sqrt{5} \\ 3\sqrt{2} \end{pmatrix}$. which of the following is equal to the magnitude of $v_1 - v_2$
   (a) $\sqrt{77}$ (b) $\sqrt{13}$ (c) $\sqrt{45}$ (d) $\sqrt{38}$

3. Let $\vec{u} = xi + 3j$ and $\vec{v} = 2i + xj$ be two vectors on the same plane. If $\vec{u} - \vec{v}$ is a unit vector ,then what is the value of *x?*
   (a) 0 or 1 (b) 1 or 2 (c) 2 or 3 (d) 3 or 4

4. Which of the following is not a unit vector?
   (a) i (c) $\frac{-i}{2} + \frac{\sqrt{3}}{2}j$
   (b) $\frac{i}{\sqrt{2}} - \frac{j}{\sqrt{2}}$ (d) $\frac{i}{5} + \frac{4j}{5}$

5. Suppose $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, such that $|\vec{u}| = 5$. Which of the following can not be equal to $\vec{u}$?
   (a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$
   (b) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (d) $\begin{pmatrix} -2\sqrt{5} \\ \sqrt{5} \end{pmatrix}$

Choose the best answer

1. Which of the following is not true?
   (a) Under rotation which is not the identity transformation exactly one point is fixed.

(b) If a non-identity transformation fixes two distinct points, then the transformation is a reflection.

(c) If a non-identity transformation fixes any point, then the transformation is not translation.

(d) If a translation T sends the origin to $P(1, 2)$, then the image of the line $y = x + 2$ is fixed.

2. Which of the following motion is not an identity transformation?

(a) A 1080° rotation counter clock wise about the point $(2, -9)$

(b) A translation T_1 followed by a translation T_2 such that $T_1((x, y)) = (x + 2, y - 5)$ and $T_2((x, y)) = (x - 2, y + 5)$

(c) A motion G such that $G((x, y)) = M(M((x, y)))$ where M is a reflection about the line: $2x + 3y - 1$

(d) Scaling transformation with scale factor 0.1

3. Let $T((x, y)) = (x + 0.5, y)$ for every (x, y), then the image of the line:

$x = 4$ under T is:

(a) $3y - 2x = 0$	(c) $x = 4.5$
(b) $y = 2x + 6$	(d) $y = 3.5$

4. Let $C((x, y)) = (x + 0.5y, y)$ for every (x, y), then the image of the line: $x = y$ under C is:

(a) $3y = 2x$	(c) $y = 6x + 1$
(b) $y = 6x$	(d) $y = 2x - 3$

5. Let $P((x, y)) = (x, 0)$ for every (x, y), then the image of the line:

6. $y = mx + b$ under P for any real number m is:

(a) $x = 0$

(b) $y = x$

(c) $y = (0. 0)$

(d) $y = 0$

7. A motion is called rigid if

(a) it sends lines into lines

(b) it sends points into points

(c) it preserves distance

(d) it sends line segments into line segments

8. If the whole plane is reflected by the line $y = x$, then the image of the line $y = 3x$ is
   (a) $y = 3x$
   (b) $y = \frac{1}{3}x$
   (c) $y = -3x$
   (d) $y = -\frac{1}{3}x$
9. The image of the line $2y + 3x = 1$ after it has been reflected by the line $y + 3x = 2$ is
   (a) $y + 18x = 17$
   (b) $y = 9x - 10$
   (c) $y = x - 3$
   (d) $2y = -11x + 9$
10. If the plane is rotated counter clock wise 90° about (b, a), then the image of (a, b) is:
   (a) $(-b, a)$
   (b) $(a - b, a + b)$
   (c) $(a, 2a - b)$
   (d) $(b - a, a - b)$
11. Which of the following statements is different from the other?
   (a) A rotation of the plane counter clock wise 360° about (8, –9)
   (b) Scaling transformation of scale factor 1.
   (c) A translation that carries the origin to (3, 2)
   (d) An identity transformation.

Choose the best answer

1. How many sides does a hexagon have?
   (a) 5 (b) 6 (c) 8 (d) 12
2. Which of the following is the sum of the interior angles of dodecagon?
   (a) 150 (b) 156 (c) 160 (d) 152
3. What is the sum of each exterior angles of a decagon?
   (a) 36 (b) 24 (c) 20 (d) 18
4. Which of the following is a regular quadrilateral?
   (a) Radius
   (b) Apothem
   (c) Tangent
   (d) Secant

5. Which of the following can NOT be used to prove that a quadrilateral is a parallelogram?
   (a) Both pairs of sides are parallel
   (b) Diagonals bisect each other
   (c) The angles add up to 360 degrees
   (d) Opposite sides are congruent.
6. What is the measure of the central angle of a regular octagon?
   (a) 30° (b) 45° (c) 60° (d) 90°
7. What is the measure of an interior angle of a 16-gon regular octagon?
   (a) 153° (b) 155.5° (c) 156° (d) 157.5°
8. Which of the following is the formula for the area of a regular polygon side whose radius is
   (a) $\frac{1}{2}nr^2\cos\frac{360^0}{n}$ (c) $\frac{1}{2}nr^2\cos\frac{180^0}{n}$
   (b) $\frac{1}{2}nr^2\sin\frac{360^0}{n}$ (d) $\frac{1}{2}nr^2\sin\frac{180^0}{n}$
9. What is the line segment from the center of a regular polygon to any of its side?
   (a) radius (c) tangent
   (b) apothem (d) secant
10. What is the area of an equilateral triangle the lengths of whose sides are 4 cm each?
   (a) $4\sqrt{3}\,\text{cm}^2$ (c) $12\sqrt{3}\,\text{cm}^2$
   (b) $8\sqrt{3}\,\text{cm}^2$ (d) $16\sqrt{3}\,\text{cm}^2$
11. What is the area of an equilateral triangle the length of whose radius is 4 cm each?
   (a) $4\sqrt{3}\,\text{cm}^2$ (c) $12\sqrt{3}\,\text{cm}^2$
   (b) $8\sqrt{3}\,\text{cm}^2$ (d) $16\sqrt{3}\,\text{cm}^2$
12. If the measure of an exterior angle a regular polygon is , how many sides does the regular polygon have?
   (a) 15 (b) 18 (c) 20 (d) 24

13. What is the area of a regular 12-side polygon which is inscribed in a circle of radius 3 cm?

(a) $9\sqrt{3}\,cm^2$ (c) $18\sqrt{3}\,cm^2$

(b) 18 cm^2 (d) 27 cm^2

14. The sum of the interior angles of a convex polygon is 306°. How many sides does it have?

(a) 15 (b) 17 (c) 19 (d) 21

15. What is the name of the intersection point of the medians of a triangle?

(a) intercept (c) incenter

(b) centroid (d) circumcenter

16. What is the radius of a regular hexagon whose area is 54?

(a) 6 cm (b) 8 cm (c) 9 cm (d) 10 cm

17. What is the area of a regular hexagon whose perimeter is 18?

(a) $27\sqrt{3}\,cm^2$ (c) $21\sqrt{3}\,cm^2$

(b) $24\sqrt{3}\,cm^2$ (d) $18\sqrt{3}\,cm^2$

18. If the interior angles of a pentagon measure 60^0, $\frac{1}{2}x^0$, $30^0 + \frac{1}{2}x^0$, $\frac{1}{2}x^0$, and $3x°$ then what is the value of ?

(a) 60 (b) 80 (c) 100 (d) 110

19. What is the perimeter of a regular 24 side polygon whose radius is 18?

(a) $240 \sin \frac{\pi}{12}$ cm (c) $120 \sin \frac{\pi}{12}$ cm

(b) $120\sqrt{3}$ cm (d) $240\sqrt{3}$ cm

20. If the length of each side of a regular polygon is and its perimeter is ,then how many sides does the polygon have?

(a) 10 (b) 20 (c) 24 (d) 32

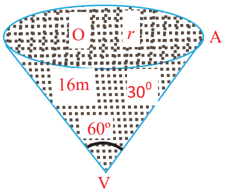
Choose the best answer

1. How long is the diagonal of a cube if its edge is 10 cm long?

(a) 5 cm (c) $10\sqrt{3}$ cm

(b) 10 cm (d) $5\sqrt{3}$ cm

2. In the following rectangular prism, if $AB = 3$ cm, $BC = 4$ cm and $CE = 5\sqrt{3}$ cm long. What is $m(\angle CAE)$?
   (a) 60º (b) 45º (c) 30º (d) 90º
3. A right circular cone has radius of the base 3cm and height 4cm. What is the surface area and volume of the cone?
   (a) $12\,\pi$ cm^3 (c) $24\,\pi$ cm^3
   (b) $18\,\pi$ cm^3 (d) $8\,\pi$ cm^3
4. In a regular hexagonal pyramid of length of base 8 cm and lateral edge 10 cm. Which one is not true?
   (a) Lateral surface area is $96\sqrt{21}$ cm^2
   (b) Total surface area is $96\sqrt{51}$ cm^2
   (c) Volume is $192\sqrt{3}$ cm^3
   (d) Base area is $96\sqrt{3}$ cm^2
5. A right circular cone has area of the base 10π cm^2 and height 5 cm. What is the height of the cone if the base area is changed to 8π cm^2 and the volume remain the same?
   (a) $\frac{25}{4}$ cm (c) $\frac{50}{4}$ cm
   (b) $\frac{15}{4}$ cm (d) $\frac{50}{3}$ cm
6. The height of a right circular cone is 16 cm with a semi – vertical angle 30º. What is the volume of the solid?
   (a) 128π cm^3
   (b) 64π cm^3
   (c) 256π cm^3
   (d) 32π cm^3



7. A prism and a pyramid have the same altitude. If the area of the base of the pyramid is twice the area of the base of the prism, What is the ratio of the volume of the pyramid to the volume of the prism?
   (a) $\frac{4}{9}$ (b) $\frac{2}{3}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{16}{81}$

8. The volume of circular cylinder of radius r and height h is
   (a) $2\pi rh^2$ (b) $2\pi r^2h$ (c) πr^2h (d) πrh

9. The volume of a right rectangular prism of dimensions w, h and ℓ is
   (a) $w\,h\,\ell$ (c) $w\,h\,\ell^2$
   (b) $\frac{1}{2}w\ell h$ (d) $w^2h^2\ell^2$

10. The volume of a circular cone whose height h and radius r is
   (a) $\frac{1}{2}\pi r^2h$ (c) $\frac{1}{2}\pi rh$
   (b) $\frac{1}{3}\pi r^2h$ (d) $\frac{1}{3}\pi rh^2$

11. The volume of a pyramid with base area A and height h is:
   (a) $\frac{1}{2}Ah^2$ (b) $\frac{1}{3}Ah^2$ (c) $\frac{1}{3}Ah$ (d) $\frac{1}{2}Ah$

12. The lateral surface area of a cylinder with height h and radius r is
   (a) $2\pi r^2$ (b) $2\pi rh$ (c) $2\pi r^2$ (d) πhr

13. The lateral surface area of a cone of radius r and height h is
   (a) $\pi r\sqrt{r^2 + h^2}$ (c) $\frac{1}{3}\pi r^2h$
   (b) $\pi r^2 + \pi r\sqrt{r^2 + h^2}$ (d) $2\pi rh$

14. The total surface area of a cylinder whose height is h and radius r is:
   (a) $\pi r(h + 2r)$ (c) $\pi r(2h + r)$
   (b) $2\pi r(h + r)$ (d) $\pi rh(r + h)$

15. The volume of a cylinder of height $2r$ and radius r is 54π cm^3. Then its height is
   (a) 6 cm (b) 0.3 cm (c) 60 cm (d) 30 cm

16. The area of the base of a pyramid is 20 cm^2. If the altitude of the pyramid is 3 cm, then its volume is
   (a) 60 cm^3 (b) 20 cm^3 (c) 30 cm^3 (d) 40 cm^3

17. The volume of a pyramid is 42 cm^3. If the altitude of the pyramid is 14 cm, then the area of its base is
   (a) 9 cm^2 (b) 1 cm^2 (c) 3 cm^2 (d) 6 cm^2

18. The volume of a right triangular pyramid whose bases have lengths 16 cm, 10 cm and an included angle 60° and height of the pyramid is 6cm is

    (a) $120\sqrt{3}$ cm^3 (c) 40 cm^3

    (b) $80\sqrt{3}$ cm^3 (d) 240 cm^3

19. The area of the base of a right prism is 100 cm^2. If its altitude measures 5 cm, then its volume is

    (a) 20 cm^3 (c) 0.05 cm^3

    (b) 500 cm^3 (d) 250 cm^3

20. The lengths of the base of a right triangular prism are 6, 8 and 10 units long. If the volume is 96 cubic units then the surface area is ________ square units.

    (a) 144 (b) 336 (c) 96 (d) 120

21. If the edge of a cube is x, then its volume and surface area are respectively.

    (a) x^3 and x^2 (c) x^3 and $2x^2$

    (b) x^3 and $4x^2$ (d) x^3 and $6x^2$

22. The diagonal of the base of a cube is $x\sqrt{2}$, then its volume is:

    (a) $2\sqrt{2}x^3$ (b) $8x^3$ (c) $2x^3$ (d) x^3

23. A cube of side 3 cm is removed from a right rectangular prism whose dimensions are 7 cm, 3 cm and 4 cm. Then the volume of remaining solid is:

    (a) 54 cm^3 (b) 27 cm^3 (c) 84 cm^3 (d) 57 cm^3

24. If a right circular cone has base radius 5cm and slanted height 8 cm, then the volume of this cone is equal to:

    (a) $\frac{25\sqrt{39}}{3}\pi \text{cm}^3$ (c) 200 πcm^3

    (b) $\frac{65\sqrt{39}}{3}\pi \text{cm}^3$ (d) $\frac{200}{3}\pi \text{cm}^3$

Choose the best answer

1. In how many ways can a committee of 3 member be formed from 7 candidates?

    (a) 7 (c) 28

    (b) 21 (d) 35

2. The probability that an electronic device produced by a company does not function properly is equal to 0.1.If 2 devices are bought ,then what is the probability that at least 1 device function properly?
   (a) 0.81 (b) 0.09 (c) 0.18 (d) 0.99
3. A team of 10 researchers consists 4 biologists and 6 chemists. If 3 person are chosen randomly from the team ,what is the probability that at least one is a biologist?
   (a) $\frac{2}{3}$ (b) $\frac{2}{5}$ (c) $\frac{5}{6}$ (d) $\frac{7}{10}$
4. Let A and B be two events. Suppose that the probability that neither event occurs is $\frac{3}{8}$. What is the probability that at least one of the events occur?
   (a) $1/8$ (b) $1/4$ (c) $5/8$ (d) $3/4$
5. A Private college has 1000 students. 60% of these students are males 45% of these students pay their payment by credit card including 175 females. What is the probability that the student is a male or credit card user?
   (a) 0.675 (c) 0.225
   (b) 0.775 (d) 0.325
6. 40% of books sold at a shop found to be Mathematics books. If three books are randomly selected from the books that are sold, what is the probability that exactly two are Mathematics books?
   (a) 0.096 (c) 0.144
   (b) 0.064 (d) 0.904
7. The numbers 1 to 19 are placed in a hat, and a number is selected, what is the probability that the number is 10 given that the number selected is known to be even?
   (a) $\frac{1}{19}$ (b) $\frac{9}{19}$ (c) $\frac{2}{9}$ (d) $\frac{1}{9}$
8. Which of the following is not true about probability of an event E?
   (a) If the probability of an event E is p, then $0 \leq p \leq 1$
   (b) Probability of a sure or a certain event is 1.
   (c) Probability of an impossible event is 0
   (d) None of the above

9. In a jar there are three white, four yellow and two blue balls all the same except colour. Three balls are drawn at random, what is the probability that at least one is blue?

   (a) $\frac{2}{9}$ (b) $\frac{7}{12}$ (c) $\frac{4}{7}$ (d) $\frac{1}{2}$

10. The probability that a red ball can be drawn is $\frac{2}{5}$ and the probability that a misael will hit its target is $\frac{5}{6}$, what is the probability that at least a red ball will be drawn or a misael hit its target?

    (a) $\frac{37}{30}$ (b) 0.9 (c) $\frac{1}{3}$ (d) $\frac{13}{30}$

11. A two digit number is made by choosing two numbers from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }. A number is selected once to make the number. If a number is choosen at random from the set of the numbers being made, what is the probability that it is an even multiple of 7?

    (a) $\frac{2}{15}$ (b) $\frac{5}{81}$ (c) $\frac{1}{11}$ (d) $\frac{7}{81}$

12. In a group of 50 students all of who are studying mathematics or computer or both 25 are studding computer and 35 are studding mathematics. Then which of the following is not true if a student is choosen at random?

    (a) The probability that he is studding computer is 0.5
    (b) The probability that he is studding mathematics is 0.7
    (c) The probability that he is studding computer but not mathematics is 0.3
    (d) The probability that he is studding mathematics and computer is 0.1

13. Three cards are drawn from a pack of fifty two playing cards. What is the probability that exactly two are black?

    (a) $\frac{13}{34}$ (b) $\frac{2}{13}$ (c) $\frac{26}{51}$ (d) $\frac{2}{51}$

14. A team of five people is choose at random from a group of four men and six women. Then what is the probability that there are at most two women in the team?

    (a) $\frac{1}{4}$ (b) $\frac{11}{42}$ (c) $\frac{3}{5}$ (d) $\frac{1}{7}$

15. A fair six side die is tossed. What is the probability of getting a number which is not a factor of 6?

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{5}$

16. The numbers 1, 2, 3, 4, 5, 6, and 7 are written on 7 identical cards. If three cards are choosen at random, what is the probability that at least one of them shows an odd number?

(a) $\frac{4}{7}$ (b) $\frac{34}{35}$ (c) $\frac{12}{35}$ (d) $\frac{6}{7}$

17. In a class a committee of 12 members is formed. Four students are choosen at random from the committee. What is the probability that at least one is a boy if 7 of them are girls?

(a) $\frac{7}{99}$ (b) $\frac{97}{99}$ (c) $\frac{28}{99}$ (d) $\frac{92}{99}$

18. A committee has 12 members out of which 5 are females. If three persons are chosen at random, then which of the following is not true?

(a) The probability that all are males is $\frac{7}{44}$

(b) The probability that exactly two are females is $\frac{7}{22}$

(c) The probability that at most two are males is $\frac{35}{44}$

(d) The probability that at least one is female is $\frac{37}{44}$

19. A box contains eight balls out of which five are red and three are white. If two balls are randomly taken out of the box, then which of the following is not true?

(a) The probability that both will be red is $\frac{5}{14}$

(b) The probability that the balls will have the same colour is $\frac{1}{2}$

(c) The probability that the balls will have different colours is $\frac{15}{28}$

(d) The probability that at least one of the ball is red is $\frac{25}{28}$

20. What is the probability that three letters choosen at random from the letters IDENTITIES contain at least one I?

(a) $\frac{17}{37}$ (b) $\frac{17}{24}$ (c) $\frac{3}{10}$ (d) $\frac{7}{10}$

Choose the best answer

1. $\log_x y = 0.2$ and $\log_a x = 1.01$. Then the value of $\log_a \frac{y^2 x^3}{a}$ is :

(a) 3.43 (b) 4.43 (c) 2.43 (d) 5.43

2. Which one of the following is not true?

(a) If $y = 10^{3x}$, then $x = \frac{1}{3} \log y$

(b) If $y = (4x-1)^{\frac{2}{3}}$; $x > \frac{1}{4}$, then $x = \frac{y\sqrt{y}+1}{4}$

(c) If $y = \sqrt{2^{\frac{-x}{3}}}$, then $x = -6 \log_2 y$

(d) If $y = \log x - \log (x + 1)$, then $x = \frac{y}{y-1}$

3. Let $y = 2^{3x}$. Then which one of the following is true?

(a) $x = \log_2 9y$ (c) $x = \log_2 \frac{y}{3}$

(b) $x = \log_8 y$ (d) $x = \log_2 \frac{3}{y}$

4. Let $y = 2^{3x}$. Then which one of the following is true?

(a) $x = \log_2 9y$ (c) $x = \log_2 \frac{y}{3}$

(b) $x = \log_8 y$ (d) $x = \log_2 \frac{3}{y}$

5. If $3 \log y - 2 \log x = \log b - \log a$, then which one of the following is true.

(a) $ay^3 = bx^2$ (c) $y = ax^b$

(b) $y = a + b^x$ (d) $y^a = b^x$

6. Let $y = 2^{(4x-1)}$. Then x is equal to:

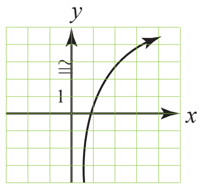
(a) $\log_{16} 2y$
(b) $\log_2(4y+1)$
(c) $\log_2(4y+1)$
(d) $\log_{16} y$

7. Let $a > 0,\ b > 0$ and m and n are real numbers. Which one of the following is correct?

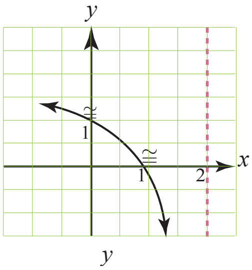
(a) $a^m \times a^m = a^{m^2}$
(b) $(a^m)^m = a^{m^2}$
(c) $a^n b^n = (ab)^{2n}$
(d) $\dfrac{a^{-m}}{b^{-n}} = \dfrac{a^n}{b^m}$

8. Which one of the following curves may represent the function $f(x) = \log_2(2-x)$?

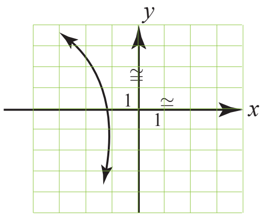
A



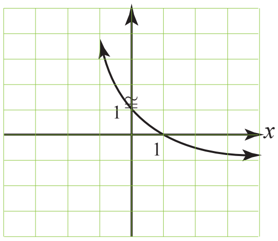
B



C



D



9. If $3^x = 2^{xy}$, $x, y > 0$; then which of the following is true?

(a) $y = \log_2 3$
(b) $x = \log_3 y$
(c) $2^y = 3^{x^2}$
(d) $2x = 3y$

10. If $\log_2 x + 3\log_2 y = 4$, then which one of the following is not true?

(a) $xy^3 = 16$
(b) $y = 2$ when $x = 2$
(c) $y = 4$ when $x = \dfrac{1}{4}$
(d) $x + 3y = 4$

11. $\log 6 - 2\log 3 + \dfrac{3}{2}\log 9$ is equal to:

(a) log 6
(b) log 9
(c) log 18
(d) log 27

12. The value of $\log_{0.04} 5$ is

(a) $\frac{-1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{5}$ (d) –0.25

13. What is the solution set of the equation $\log_2 (x + 3) + \log_2 (3x + 1) = 7$?

(a) {5} (b) $\left\{\frac{1}{3}\right\}$ (c) {–3} (d) {}

14. What is the solution set of the equation $\log (10x + 1) + \log(2.5x - 2) = 2$?

(a) {2.8} (b) {2.4} (c) {5} (d) {–0.1}

15. What is the solution set of the equation $\log (2x^2 - 3x + 8) = 1$?

(a) {–2,2} (c) {1,0}

(b) {0.25,–0.5} (d) {2,–0.5}

16. What is the solution set of the equation

$\log 5 - \log(2x+1) = \log 6 - \log(3x+1)$?

(a) $\left\{\frac{5}{6}\right\}$ (b) $\left\{\frac{1}{2}\right\}$ (c) $\left\{\frac{1}{3}\right\}$ (d) $\left\{\frac{5}{4}\right\}$

17. The solution set of $\log_2 \left[2 + \log_3 (x+3)\right] = 0$ is equal to:

(a) $\left\{-\frac{8}{3}\right\}$ (b) $\left\{\frac{1}{3}\right\}$ (c) { } (d) {–1}

Choose the best answer

1. Which of the following is equal to $\lim_{x \to 3} (x^2 - 5x + 2)$?

(a) 16 (b) –4 (c) 2 (d) –3

2. Which of the following is equal to $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$?

(a) 0 (c) –2

(b) 1 (d) The limit does not exist

3. $\lim_{x \to 0} \frac{x^3 - x}{4x - x^2}$ is equal to:

(a) $-\frac{1}{4}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

4. $\lim_{x \to 0} (\sin(2x) - 1)$ is equal to:

(a) 1 (b) 0 (c) −1 (d) -2

5. $\lim_{x \to 0} \frac{\sin(3x)}{2x}$ is equal to:

(a) $\frac{1}{2}$ (b) 1 (c) 3 (d) $\frac{3}{2}$

6. What is the value of $\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 16}$?

(a) $\frac{3}{8}$ (b) $-\frac{1}{4}$ (c) $\frac{5}{16}$ (d) −1

7. $\lim_{x \to 0} \left(1 - \frac{2}{x}\right)^{1-3x}$ is equal to:

(a) e^{-2} (b) $\sqrt{e}$ (c) e^{-5} (d) e^{6}

8. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ is equal to:

(a) $\frac{1}{4}$ (b) 1 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

9. What is the value of $\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$?

(a) $\frac{1}{3}$ (c) $\frac{1}{6}$

(b) $-\frac{1}{2}$ (d) The limit does not exist

10. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$ is equal to:

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$

11. $\lim_{x \to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$ is equal to:

(a) 2 (b) 4 (c) 6 (d) 8

12. Let $f(x) = x^2 - 4$. Which of the following is the difference quotient of f at $x = 1$ for $h > 0$?

(a) $h^2 - 4x$ (c) $(h + 1)^2 - 3$

(b) $2x + h$ (d) $(h + 1)^2 - x^2 + 4$

13. Let $f(x)=2x^2-3x+7$. Which of the following is equal to $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$?

(a) $4x-3$ (c) 9

(b) $4x-3h$ (d) 15

14. Which of the following is the slope of the line tangent to the graph of $f(x)=x^5-4x^3+1x=-2$?

(a) 33 (b) 1 (c) 129 (d) 0

15. Which of the following is the equation of the line tangent to the graph of $f(x)=\frac{3}{2}x^2-4x+1$?

(a) $y=5x-1$ (c) $y=\frac{5}{2}(2x-5)$

(b) $y=3x+4$ (d) $y=\frac{3}{4}(4x+2)$

16. Which of the following is the derivative of $f(x)=1+x-\sin x$?

(a) $1-\cos x$ (c) $-\cos x$

(b) $1+\cos x$ (d) $\sin x$

17. Which of the following is the slope of the line tangent to the graph of $f(x)=\sin x-\cos x+2$ at?

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{\sqrt{2}}{2}$ (d) $\sqrt{2}$

18. Which of the following is the equation of the line tangent to the graph of $f(x)=1-3\cos x$ at $x=0$?

(a) $y=4x+2$ (c) $y=1$

(b) $y=-2$ (d) $y=-2x+2$

19. Which of the following is the equation of the line tangent to the graph of $f(x)=x-\sin x$ at $x=0$?

(a) $y=0$ (b) $x=2$ (c) $y=x$ (d) $y=-x$

20. Let $f(x)=\frac{1}{x+2}$. Which of the following is equal to $\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$?

(a) $\frac{1}{2}$ (c) -1

(b) $\frac{1}{3}$ (d) The limit does not exist

21. Which of the following is equal to $\int (3x^3 - 1)\, dx$

(a) $12x^4 - x + c$

(b) $\frac{3}{4}x^4 - x + c$

(c) $9x^2 + c$

(d) $9x^2 - x + c$

22. Which of the following is equal to $\int 2 \sin x\, dx$

(a) $-\frac{1}{2}\cos x + c$

(b) $2x + \cos x + c$

(c) $-\frac{1}{2}x - \cos x + c$

(d) $-2 \cos x + c$

23. What is the area of the region bounded by the graph of $f(x) = 3x^2$ and the axis between $x = -1$ and $x = 2$?

(a) 27 square units

(b) 13 square units

(c) 9 square units

(d) 7 square units

24. Which of the following is the area of the region bounded by the graph of $f(x) = 5x - x^2$ and the x?

(a) $\frac{125}{6}$ square units

(b) $\frac{25}{3}$ square units

(c) $\frac{25}{6}$ square units

(d) $\frac{125}{2}$ square units

25. What is the area of the region bounded by the graph of and the axis between $x = 0$ and $x = \frac{\pi}{2}$?

(a) $\frac{\sqrt{2}}{2}$ square units

(b) 1 square unit

(c) $\frac{1}{2}$ square units

(d) $\sqrt{2}$ square units

Now answer the following questions

1. Evaluate, correct, to **four** significant figures, (573.06×184.25)

(a) 105500.00 (c) 105600.00

(b) 105622.00 (d) 105532.00

2. Change 432_{five} to a number in base there.

(a) 10100_{three} (c) 11101_{three}

(b) 10110_{three} (d) 111000_{three}

3. Given that A and B are sets, $n(A) = 8$, $n(B) = 12$ and $n(A \cap B) = 3$, find $n(A \cup B)$.

(a) 17 (b) 15 (c) 20 (d) 23

4. If $\sqrt{24} + \sqrt{96} - \sqrt{600} = y\sqrt{6}$, find the value of y.

(a) 4 (b) –4 (c) –2 (d) 2

5. Evaluate $23 \times 54 \pmod 7$.

(a) 3 (b) 2 (c) 5 (d) 6

6. If $4^{3x} = 16^{x+1}$, find the value of x.

(a) –2 (b) 1 (c) 2 (d) –1

7. A weaver bought a bundle of grass for L$ 50.00 from which he made 8 mats. If each mat was sold for L$ 15.00, find the percentage profit.

(a) 40% (b) 120% (c) 140% (d) 240%

8. Find the 17th term of the Arithmetic Progression (A.P): –6, –1, 4, …

(a) 79 (b) 74 (c) –86 (d) –91

9. M varies directly as n and inversely as the square of p. If $M = 3$ when $n = 2$ and $p = 1$, find M in terms of n and p.

(a) $M = \dfrac{3n^2}{2p^2}$ (c) $M = \dfrac{2n}{3p^2}$

(b) $M = \dfrac{2n}{3p}$ (d) $M = \dfrac{3n}{2p^2}$

10. If $a = 3$ and $b = -7$, find the value of $\dfrac{5b + (a+b)^2}{(a-b)^2}$

(a) –0.51 (b) –0.19 (c) 0.19 (d) 0.51

11. Three boys shared L$ 10,500.00 in the ratio 6: 7: 8. Find the largest share.
    (a) L$ 3,500.00 (c) L$ 5, 000.00
    (b) L$ 4,500.00 (d) L$ 4, 000.00

12. The length of a piece of stick is 1.75 m. A boy measured it as 1.80m. Find the percentage error.
    (a) $4\frac{7}{9}\%$ (c) $4\frac{6}{7}\%$
    (b) $2\frac{7}{9}\%$ (d) $4\frac{4}{7}\%$

13. If $5x + 3y = 4$ and $5x - 3y = 2$, what is the value of $(25x^2 - 9y^2)$.
    (a) 8 (b) 2 (c) 16 (d) 20

14. Mary has L$ 3.00 more than Ben but L$ 5.00 less than Jane. If Mary has L$ x, how much does Jane and Ben have altogether?
    (a) $2x + 2$ (c) $2x + 8$
    (b) $2x - 2$ (d) $2x - 8$

15. Consider the statements:
    p: Stephen is intelligent;
    q: Stephen is good in Mathematics.
    If $p \Rightarrow q$, which of the following is a valid conclusion?
    (a) If Stephen is not good at Mathematics, then he is intelligent.
    (b) If Stephen is not intelligent, then he is not good at Mathematics.
    (c) If Stephen is not good at Mathematics, then he is not intelligent.
    (d) If Stephen is good at Mathematics, then he is intelligent.

16. What value of p will make $(x^2 - 4x + p)$ a perfect square?
    (a) -2 (b) 16 (c) -8 (d) 4

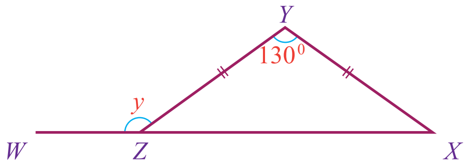
17. Find the value of x such that $\frac{1}{x} + \frac{4}{3x} - \frac{5}{6x} + 1$ is zero.
    (a) $-\frac{3}{2}$ (c) $\frac{1}{6}$
    (b) $\frac{1}{4}$ (d) $-\frac{7}{6}$

18. Make t the subject of $k = m\sqrt{\dfrac{t+p}{r}}$

(a) $t = \dfrac{rk^2 + p}{m^2}$ (c) $t = \dfrac{rk^2 - p}{m^2}$

(b) $t = \dfrac{rk^2 + p^2}{m^2}$ (d) $t = \dfrac{rk^2 + pm^2}{m^2}$

19.



In the diagram, $|\overline{XY}| = |\overline{YZ}|$ and $\angle XYZ = 130^{\circ}$. Find the value of y.

(a) 155° (c) 25°

(b) 65° (d) 50°

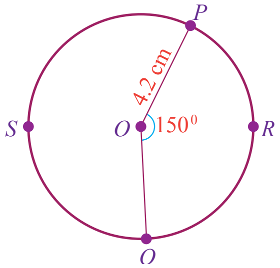
20. An exterior angle of a regular polygon is 22.5°. Find the number of sides.

(a) 13

(b) 16

(c) 15

(d) 14



In the diagram, $\angle POQ = 150°$ and the radius of the circle, $PSQR$ is 4.2cm. [Take $\pi = \dfrac{22}{7}$] Use the information to answer questions 21 and 22.

21. Find the length of the minor are PRQ.

(a) 15.4 cm (c) 17.64 cm

(b) 11.0 cm (d) 23.1 cm

22. Find the area of the sector $OPSQ$.

(a) 15.4 cm^2 (c) 32.34 cm^2

(b) 17.64 cm^2 (d) 23.1 cm^2

23. A ladder 6m long leans against a vertical wall at an angle 53° to the horizontal. How high up the wall does the ladder reach?

(a) 3.611 m (c) 4.521 m

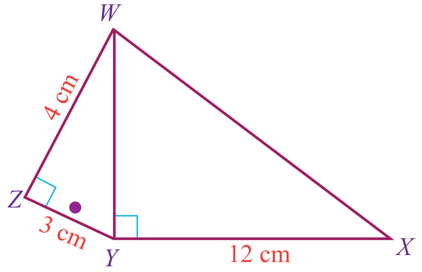
(b) 4.792 m (d) 3.962 m

24. A cylinder opened at one end has a radius of 3.5 cm and height 8 cm. Calculate the total surface area. $\left[\text{Take } \pi = \frac{22}{7}\right]$

(a) 214.5 cm^2 (c) 165.0 cm^2

(b) 212.0 cm^2 (d) 126.5 cm^2

25.



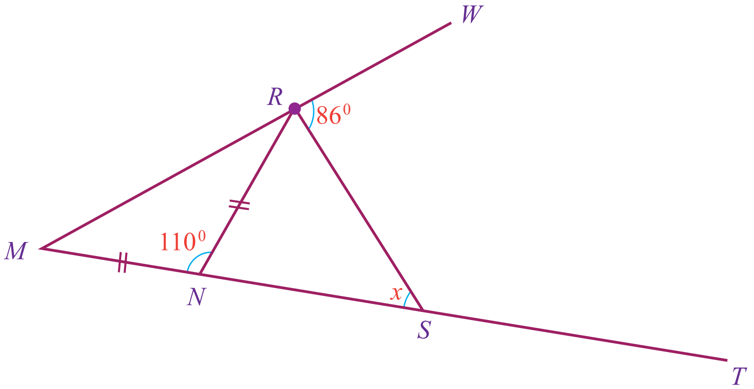
In the diagram $\angle WZY$ and $\angle WYX$ are right angles. Find the perimeter of $WXYZ$.

(a) 32 cm (b) 30 cm (c) 35 cm (d) 37 cm

26. The length of a rectangle is 10 cm. If its perimeter is 28 cm, find the area.

(a) 30 cm^2 (b) 60 cm^2 (c) 40 cm^2 (d) 80 cm^2

27.



In the diagram, MRW and $MNST$ are straight lines. $|\overline{MN}| = |\overline{NR}|$, $\angle MNR = 110^o$ and $\angle WRS = 86°$, find the value of x.

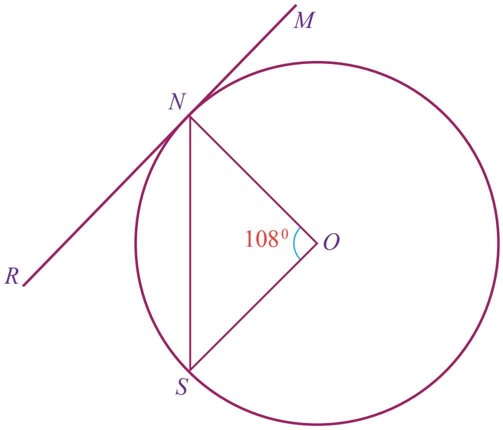
(a) 51° (b) 70° (c) 42° (d) 86°

28. A boy 1.4 m tall, stood 10m away from a tree of height 12 m. Calculate, correct to the nearest degree, the angle of elevation of the top of the tree from the boy's eyes.

(a) 47° (b) 71° (c) 19° (d) 8°

29. Given that $\sin(5x-28)° = \cos(3x-50)°$, $0° \le x \le 90°$, find the value of x.

(a) 39
(b) 21
(c) 32
(d) 14



In the diagram, MNR is a tangent to the circle center O at N and $\angle NOS = 108^{\circ}$. Use the information to answer questions 30 and 31.

30. Find $\angle OSN$.

(a) 72° (b) 42° (c) 18° (d) 36°

31. Find $\angle SNR$.

(a) 54° (b) 42° (c) 36° (d) 72°

32. Mrs Gabriel is pregnant. The probability that she will give birth to a girl is $\frac{1}{2}$ and the probability that the baby will have blue eyes is $\frac{1}{4}$. What is the probability that she will give birth to a girl with blue eyes?

(a) 1 (c) $\frac{3}{4}$

(b) $\frac{1}{8}$ (d) $\frac{1}{4}$

33. The mean of a set of 10 numbers is 56. If the mean of the first nine numbers if 55, find the 10th number.

(a) 75 (c) 55

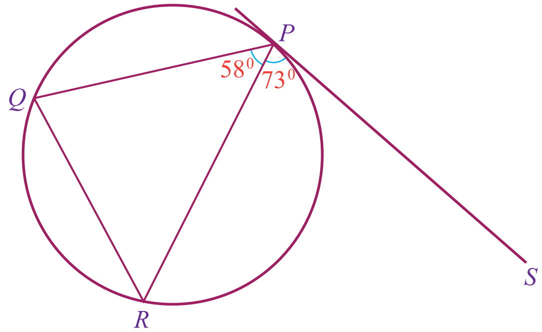
(b) 45 (d) 65

34. Simplify: $\frac{2-18m^2}{1+3m}$

(a) $2(1 + 3m)$ (c) $2(1 - 3m^2)$

(b) $2(1 + 3m^2)$ (d) $2(1 - 3m)$

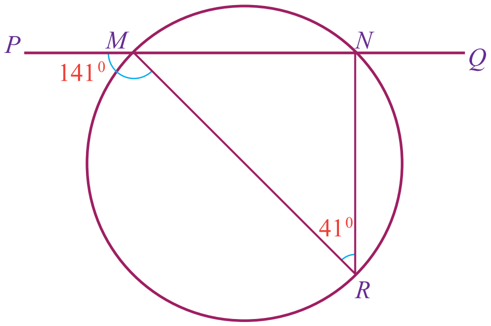
35.



The diagram shows triangle PQR is inscribed in a circle. $\overline{PS}$ is a tangent to the circle at P. Find $\angle PRQ$.

(a) 131° (b) 58° (c) 73° (d) 49°

36.



In the diagram triangle MNR is inscribed in circle. MNR and $\overline{PQ}$ is a straight line. If $\angle MRN = 41°$ and $\angle PMR = 141°$, find $\angle QNR$.

(a) 39° (b) 141° (c) 110° (d) 80°

37. Solve: $\frac{y+2}{4} - \frac{y-1}{3} > 1$.

(a) $y < -2$ (b) $y < -10$ (c) $y < 2$ (d) $y < 10$

The ages (years) of some members in a singing group are: 12, 47, 49, 15, 43, 41, 13, 39, 43, 41, and 36. Use the information to answer questions 38 and 39.

38. Find the lower quartile.

(a) 12 (b) 13 (c) 30 (d) 15

39. Find the mean

(a) 35.54 (b) 33.35 (c) 34.45 (d) 36.44

40. Find, correct to two decimal places, the volume of a sphere whose radius is 3 cm.

$\left[\text{Take}\pi = \frac{22}{7}\right]$

(a) 113.14 cm^3 (c) 105.29 cm^3

(b) 88.12 cm^3 (d) 72.57 cm^3

41. The lengths of the parallels ides of a trapezium are 9 cm and 12 cm. If the area of the trapezium is 105 cm^2, find the perpendicular distance between the parallel sides.

(a) 5 cm (c) 7 cm

(b) 10 cm (d) 15 cm

42. Find the volume of a cone of radius 3.5 cm and vertical height 12 cm.

$\left[\text{Take}\pi = \frac{22}{7}\right].$

(a) 154.0 cm^3 (c) 142.0 cm^3

(b) 21.0 cm^3 (d) 15.5 cm^3

43. A local community has two newspapers; the Morning Times and the evening Dispatch. The Morning Times is read by 45% of households and the Evening Dispatch by 60%. If 20% of the households read both papers, find the probability that a particular household reads at least one paper?

(a) 0.45 (c) 0.95

(b) 0.65 (d) 0.85

44. A rectangle has width $\frac{3}{4}$ cm and an area $3\frac{3}{8}$ cm^2. Find the length.

(a) 6 cm (c) $4\frac{1}{2}$ cm

(b) $2\frac{5}{8}$ cm (d) 12 cm

45. The mean of two numbers x and y is 4. Find the mean for the four numbers x, $2x$, y and $2y$.

(a) 6 (b) 4 (c) 2 (d) 8

46. The straight line $y = mx - 4$ passes through the pint (–4, 16). Calculate the gradient of the line.

(a) 5 (c) 3
(b) –3 (d) –5

47. If the equations $x^2 - 5x + 6 = 0$ and $x^2 + ax + 6 = 0$, have common roots, find the value of a.

(a) –5 (b) 6 (c) –6 (d) 5

48. A trader made a loos of 15% when an article was sold. Find the ratio fo the selling price to the cost price.

(a) 3 : 20 (c) 20 : 23
(b) 3 : 17 (d) 17 : 20

49. Given that $\log_3 27 = 2x + 1$, find the value of x.

(a) 1 (b) 0 (c) 2 (d) 3

50. Solve: $6x^2 = 5x - 1$

(a) $\{2, 3\}$ (c) $\left\{\frac{1}{2}, -\frac{1}{3}\right\}$
(b) $\{0, 3\}$ (d) $\left\{\frac{1}{2}, \frac{1}{3}\right\}$

Write you name, index number, signature and date in ink in the spaces provided above.

Answer ten questions in all. All the questions in Section A and five questions from section B.

In each question, all necessary details of working, including rough work, must be shown with the answer.

Give answers as accurately as data and tables allow.

A graph sheet is provided for your use on page 16 of your booklet.

The use of non-programmable, silent and cordless calculator is allowed.

Write in the space provided below, the question numbers of the questions you have answered, in the order in which you have written them.

Answer all the questions in this section. All questions carry equal marks.

1. The terms $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}...$ form a Geometric Progression (GP)

   If the sum of the GP is $(2^{36} - 2^{-6})$, find the number of terms.

2. Given that $p = \sqrt{\frac{mx}{t} - t^2x}$

   (a) Make x the subject

   (b) If $m = 7$, $p = -3$ and $t = 4$, find the value of x.

3. The diagonal of a rectangular field is 169 m. If the ratio of the length to the width is 12 :5, find the:

   (a) Dimensions;

   (b) Perimeter of the field.

4. Two regular polygons X and Y are such that the number of sides of X is 3 more than the number of sides of Y. If the sum of the exterior angles of X and Y is 117°, how many sides has X?

5. The scores of 10 learners in a test arranged in ascending order is: 12, 13, 13, a, $(a + 3)$, $(52 - 2a)$, $(2a - 8)$, 23, 23 and 24. If the median mark is 20, find the:

   (a) value of a;

   (b) mean

Answer five questions only form this section. All questions carry equal marks.

6. (a) Copy and complete the table for $y = 2x^2 + 3x - 7$ for $-4 \leq x \leq 2$.

x	−4	−3	−2	−1	0	1	2
y		2			−7		

(b) Using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis, draw the graph of $y = 2x^2 + 3x - 7$ for $-4 \leq x \leq 2$.

(c) Use the graph to find the roots of:

(i) $2x^2 + 3x - 7 = 0$

(ii) $2x^2 + 5x - 4 = 0$

7. (a) Three friends shared an amount in the ratio $\frac{2}{3} : \frac{3}{5} : \frac{1}{4}$. If the largest share was L\$ 12,000.00, how much was shared between them?

(b) A shopkeeper gave a bill to a customer. The bill was in two digits but was mistakenly interchanged thereby undercharging the customer by L\$ 45.00. If the sum of the digits is 9, find the actual value of the bill.

8. (a) The lengths of twelve pols form an Arithmetic Progression (A.P). If the third pole is 3 m and the eighth pole is 5 m, find the:

(i) length of the first pole;

(ii) sum of the lengths of the poles.

(b) The cost of three chairs and four tables is L\$ 320.00. If two chairs and five tables of same kind cost L\$ 330,00, find the cost of:

(i) 4 chairs; (ii) 3 tables;

9.

(a) Copy and complete the table of values for $y = 2\cos x + \sin x$, $0° \leq x \leq 360°$.

y	0°	45°	90°	135°	180°	225°	270°	315°	360°
x	2.0		1.0		−2.0		−1.0		

(b) Using a scale of 2 cm to 45° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 2\cos x + \sin x$ for $0^{o} \leq x \leq 360°$.

(c) On the same axes, draw the graph of $y = 1$.

(d) From the graph, find the:

(i) minimum value of $y = 2\cos x + \sin x$;

(ii) values of x for which $2\cos x + \sin x - 1 = 0$

10.

Height (cm)	1	2	3	4	5	6	7
No of seedlings	4	5	$x-2$	x	y	$y-12$	1

The table shows the height (cm) of 50 seedlings in a garden. Given that the mean height of the seedlings is 4 cm, find the:

(a) values of x and y; (b) median height.

11. (a) A bus has petrol tank in the shape of a cylinder. The cylinder is 125 cm long and has a diameter of 42 cm.

(i) How much will a full tank of petrol cost at the rate of L$ 5.23 per litre?

(ii) If the bus uses petrol at the rate of 12.5 litres for every 50 km, how far can the bus travel on a full tank of petrol?

(b) The interior angle of a regular polygon is x. If x is 9° less than the average of 153° and 145°, find the number of sides of the polygon.

12. (a) A bag contains 18 balls that differ only in colour; 11 are blue and 7 red. If two balls are picked, one after the other without replacement, find the probability that both are:

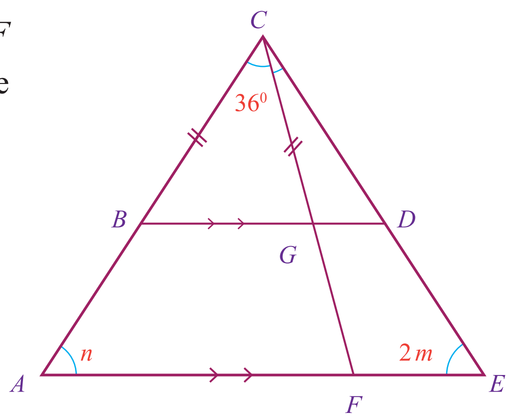
(i) blue; (iii) of different colours.

(ii) of same colour,

13. (a) The radius and slant height of a right circular cone are 9 cm and 15 cm respectively. Find, correct to one decimal place, the:

(i) height; (ii) volume of the cone.

(b) In the diagram, $\overline{BD} \| \overline{AE}$. If $\angle ACF = 360°$ and $|BC| = |CG|$, find the values of m and n.



Health Related Caution

What are the ways to avoid dengue and malaria fever?

- Time your outings.
- Reduce mosquito habitat.
- Sleep under mosquito-net.
- Put screens on windows and doors.
- Keep your house airy and well-lit.
- Do not let water stagnate anywhere.
- Wear long pants and long sleeves to cover your body.
- Apply mosquito repellent with DEET (diethyltoluamide) to exposed skin.
- Treat clothing, mosquito nets, tents, sleeping bags and other fabrics with an insect repellent called permethrin.

How can a person reduce the risk of getting HIV?

- Get tested for HIV.
- Do not inject drugs.
- Choose less risky sexual behaviors.
- Use condoms every time you have sex.
- Limit your number of sexual partners.
- Get tested and treated for STDs.
- Talk to your health care provider about pre-exposure prophylaxis (PrEP).

What is Bullying?

Any unwanted written, verbal, graphic, or physical act by an individual or group toward another person(s) that causes harm or distress.

Types of Bullying

- Physical
- Verbal
- Social
- Emotional
- Cyber

Signs of Bullying

- Headaches
- Depression
- Loss of friends
- School absenteeism
- Academic problems

What You Can Do

PREVENT

- Be a role model for positive communication, healthy relationships, and self-care.
- Reinforce acts of kindness, respect, and inclusion.
- Set policies and rules about bullying.

RECOGNIZE

- Know the definition of bullying and its many forms.
- Talk with and actively listen to the youth who confide in you.
- Watch for warning signs of bullying.

INTERVENE

- If you witness bullying behavior
- Respond quickly and consistently to send the message that it is not acceptable.
- Separate the students involved.
- Meet any immediate medical or mental health needs.
- Stay calm and model respectful behavior.

Source: Teacher's Diary on *Cyber-Crime Awareness* by UNODC, Cybercrime and MoE, Republic of Liberia

WHAT IS CYBERCRIME?

Cybercrime is criminal activity that either targets or uses a computer, a computer network or a networked device. Most cybercrime is committed by cybercriminals or hackers who want to make money or take advantage of a person.

Types of Cybercrime

- Email and internet fraud.
- Identity fraud (where personal information is stolen and used).
- Theft of financial or card payment data.
- Theft and sale of corporate data.
- Cyber extortion (demanding money to prevent a threatened attack).
- Ransomware attacks (a type of cyberextortion).
- Cryptojacking (where hackers mine cryptocurrency using resources they do not own).
- Cyberespionage (where hackers access government or company data).
- Interfering with systems in a way that compromises a network.
- Infringing copyright.
- Illegal gambling.
- Selling illegal items online.
- Soliciting, producing, or possessing child pornography.

How to Prevent Cyber Crimes?

- Enforce concrete security and keep it up-to-date.
- Never give out personal information to a stranger.
- Check security settings to prevent cybercrime.
- Using an antivirus software helps to recognize any threat or malware before it infects the computer system.
- When visiting unauthorized websites, keep your information secure.
- Restriction on access to your most valuable data.
- Backup all data, system, and considerations.
- Don't use free USB sticks.

Source: Teacher's Diary on *Cyber-Crime Awareness* by UNODC, Cybercrime and MoE, Republic of Liberia

REPUBLIC OF LIBERIA
MINISTRY OF EDUCATION

TEXTBOOK MANAGEMENT
Student Textbook Assignment Form

School : ____

District : ______________________________Country: ____________________________

Teacher : __________________________________Subject: _________________________

Grade : ____

No.	Name of the Student	Name of the Textbook	Barcode Number	Condition of the TB when received	Date Received	Date Returned	Condition of TB when returned	Signature of Student
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								